空間 2 次元での単純化 Keller-Segel 方程式の時間無限大での挙動

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1. Introduction

In this lecture, we consider the following Cauchy problem:
$u = u(t,x), \psi = \psi(t,x),\ t > 0, x \in \mathbb{R}^2$

$$(\mathrm{KS})_\psi \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \psi), & t > 0,\ x \in \mathbb{R}^2, \\ -\Delta \psi = u, & t > 0,\ x \in \mathbb{R}^2, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^2 \end{cases}$$

$$\psi(t,x) := (N * u)(t,x) = \int_{\mathbb{R}^2} N(x-y) u(t,y)\, dy$$

$$\nabla \psi = \nabla N * u$$

$$u(t,x) \geq 0,\ u_0(x) \geq 0, \quad t > 0,\ x \in \mathbb{R}^2$$

- A simplified version of a usual chemotaxis system by Keller and Segel ▸ parabolic system
- A model of self-attracting particles

▸ Conservation quantities

The Keller-Segel model

Keller-Segel, J. Theor. Biol., 1970
$u = u(t, x)$: the population density of amoebae at time t and position x,
$\psi = \psi(t, x)$: the concentration of a chemical attractant

$$\begin{cases} \partial_t u = \underbrace{\Delta u}_{\text{diffusion}} - \underbrace{\nabla \cdot (u \nabla \psi)}_{\text{chemotaxis}}, & t > 0,\ x \in \mathbb{R}^2, \\ \tau \partial_t \psi = \underbrace{\Delta \psi}_{\text{diffusion}} - \underbrace{a\psi}_{\text{consumption}} + \underbrace{u}_{\text{production}}, & t > 0,\ x \in \mathbb{R}^2, \end{cases}$$

where $\tau > 0$ and $a \geq 0$.
Letting $\tau \to 0$ and $a = 0$ in this system leads to $(\mathrm{KS})_\psi$.

- Basic properties of nonnegative solutions u to (KS)

1. Mass conservation law:

$$\int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx$$

▸ Proof(MCL)

2. The conservation of the center of mass:

$$\int_{\mathbb{R}^2} xu(t,x)\,dx = \int_{\mathbb{R}^2} xu_0(x)\,dx$$

▸ Proof(CCM)

3. The second Moment identity: $M := \int_{\mathbb{R}^2} u_0(x)\,dx$

$$\int_{\mathbb{R}^2} |x|^2 u(t,x)\,dx = \int_{\mathbb{R}^2} |x|^2 u_0(x)\,dx + 4M\left(1 - \frac{M}{8\pi}\right)t,$$

▸ Proof(SMI)

We prove these formally.

▸ Three cases

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla \psi) = \nabla \cdot (\nabla u - u \nabla \psi)$$

- Mass conservation law

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} u \, dx &= \int_{\mathbb{R}^2} \partial_t u \, dx \\ &= \int_{\mathbb{R}^2} \nabla \cdot (\nabla u - u \nabla \psi) \, dx \\ &= 0 \end{aligned}$$

CL

- The conservation of the center of mass: $i = 1, 2$

$$\frac{d}{dt}\int_{\mathbb{R}^2} x_i u\,dx = \int_{\mathbb{R}^2} x_i \partial_t u\,dx = \int_{\mathbb{R}^2} x_i \nabla\cdot(\nabla u - u\nabla\psi)\,dx$$

$$= -\int_{\mathbb{R}^2} \langle \nabla x_i, \nabla u - u\nabla\psi\rangle\,dx$$

$$= -\underbrace{\int_{\mathbb{R}^2} \frac{\partial u}{\partial x_i}\,dx}_{=0} + \int_{\mathbb{R}^2} u(\frac{\partial N}{\partial x_i} * u)\,dx$$

$$(\frac{\partial N}{\partial x_i} * u)(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} u(t,y)\frac{x_i - y_i}{|x-y|^2}\,dy$$

$$\int_{\mathbb{R}^2} u(\frac{\partial N}{\partial x_i} * u)\,dx = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,x)u(t,y)\frac{x_i - y_i}{|x-y|^2}\,dydx$$

Replacing x and y of the integrand on the right-hand side, we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,x)u(t,y)\frac{x_i - y_i}{|x-y|^2}\,dydx = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,y)u(t,x)\frac{y_i - x_i}{|y-x|^2}\,dxdy$$

By this,

$$\int_{\mathbb{R}^2} u(\frac{\partial N}{\partial x_i} * u)\,dx$$

$$= -\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,x)u(t,y)\frac{x_i - y_i}{|x-y|^2}\,dydx$$

$$= -\frac{1}{2\pi}\cdot\frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} u(t,y)u(t,x)\underbrace{\left(\frac{x_i - y_i}{|x-y|^2} + \frac{y_i - x_i}{|y-x|^2}\right)}_{=0} dxdy$$

$$= 0$$

Hence

$$\frac{d}{dt}\int_{\mathbb{R}^2} x_i u\,dx = 0$$

- The second moment identity

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2 u\,dx &= \int_{\mathbb{R}^2}|x|^2\partial_t u\,dx = \int_{\mathbb{R}^2}|x|^2\Delta u\,dx \\
&\quad - \int_{\mathbb{R}^2}|x|^2\nabla\cdot(u\nabla\psi)\,dx \\
&= \int_{\mathbb{R}^2}\underbrace{\Delta|x|^2}_{=4}\,u\,dx + \int_{\mathbb{R}^2}\langle\nabla|x|^2, u\nabla\psi\rangle\,dx \\
&= 4\int_{\mathbb{R}^2}u\,dx + 2\int_{\mathbb{R}^2}\langle x, u\nabla\psi\rangle\,dx
\end{aligned}
$$

$$\int_{\mathbb{R}^2} \langle x, u\nabla\psi\rangle \, dx = \frac{-1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(t,x)u(t,y) \frac{\langle x, x-y\rangle}{|x-y|^2} \, dydx$$

Replacing x and y of the integrand on the right-hand side, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(t,x)u(t,y) \frac{\langle x, x-y\rangle}{|x-y|^2} \, dydx \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(t,y)u(t,x) \frac{\langle y, y-x\rangle}{|y-x|^2} \, dxdy \end{aligned}$$

By this,

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,x)u(t,y)\frac{\langle x, x-y\rangle}{|x-y|^2}\,dydx$$
$$= \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} u(t,y)u(t,x)\underbrace{\left(\frac{\langle x, x-y\rangle}{|x-y|^2}+\frac{\langle y, y-x\rangle}{|y-x|^2}\right)}_{=1}dxdy$$
$$= \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} u(t,y)u(t,x)\,dydx = \frac{1}{2}\left(\int_{\mathbb{R}^2} u(t,x)\,dx\right)^2.$$

Hence

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2u\,dx = 4\underbrace{\int_{\mathbb{R}^2}u\,dx}_{=M} - \frac{1}{2\pi}\underbrace{\left(\int_{\mathbb{R}^2}u\,dx\right)^2}_{M^2}$$
$$= 4M\left(1-\frac{1}{8\pi}M\right).$$

Mass conservation law

$$\int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx, \quad t > 0$$

- The global existence and large-time behavior of nonnegative solutions heavily depend on the total mass $\int_{\mathbb{R}^2} u_0\,dx$:
  - Supercritical case: $\int_{\mathbb{R}^2} u_0\,dx > 8\pi$
    Finite-time blowup
  - Subcritical case: $\int_{\mathbb{R}^2} u_0\,dx < 8\pi$
    Global existence and boundedness of nonnegative solutions,
    Forward self-similar solutions
  - Critical case: $\int_{\mathbb{R}^2} u_0\,dx = 8\pi$
    Global existence of nonnegative solutions, Stationary solutions

Remark 1.1

$$(\mathrm{KS})_\psi \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \psi), & t > 0,\ x \in \mathbb{R}^2, \\ -\Delta \psi = u, & t > 0,\ x \in \mathbb{R}^2, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^2, \end{cases}$$

where

$$\psi(t,x) := (N * u)(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} u(t,y)\, dy,$$

$$\nabla \psi(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} u(t,y)\, dy$$

- $\psi(t) \in L^1_{loc}(\mathbb{R}^2),\ t > 0 \iff u(t) \log(1+|x|) \in L^1,\ t > 0$

In what follows, we consider the following Cauchy problem:
$u = u(t,x),\ t > 0, x \in \mathbb{R}^2$

$$\text{(KS)} \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u(\nabla N * u)), & t > 0,\ x \in \mathbb{R}^2, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^2. \end{cases}$$

$$N(x) := \frac{1}{2\pi} \log \frac{1}{|x|} \quad \text{(the Newtonian potential)},$$

$$\nabla N(x) = \Big(\frac{\partial N}{\partial x_1}(x), \frac{\partial N}{\partial x_2}(x) \Big) = -\frac{1}{2\pi} \frac{x}{|x|^2},$$

$$(\nabla N * u)(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} u(t,y)\, dy$$

$$u(t,x) \geq 0,\ u_0(x) \geq 0, \quad t > 0,\ x \in \mathbb{R}^2$$

The purpose of this lecture

- In the subcritical and critical cases, under a very general condition on the nonnegative initial data u_0 we discuss the following:
  - Large-time behavior of nonnegative solutions

1.1. The subcritical case $\int_{\mathbb{R}^2} u_0\,dx < 8\pi$

Global existence of nonnegative solutions

- Biler-Karch-Laurençot-Nadzieja, Math. Meth. Appl. Sci., 2006

  $u_0 \geq 0$, radial, $u_0 \in L^1$ (radial solutions)

- Blanchet-Dolbeault-Perthame, Electron. J. Differential Equations, 2006

  $u_0 \geq 0, \quad u_0, \; u_0 \log u_0, \; |x|^2 u_0 \in L^1$

- N', Differential Integral Equations, 2011
  $u_0 \geq 0, \quad u_0 \in L^1$

Notation For $1 \leq p \leq \infty$,

$L^p := L^p(\mathbb{R}^2)$: the usual Lebesgue space on $\mathbb{R}^2$ with norm $\|\cdot\|_{L^p}$

The equation in the system (KS)

$$\partial_t u = \Delta u - \nabla \cdot (u(\nabla N * u)), \quad t > 0, \ x \in \mathbb{R}^2 \qquad (1.1)$$

is invariant under the similarity transformation

$$u_\lambda(t,x) := \lambda^2 u(\lambda^2 t, \lambda x) \qquad (\lambda > 0),$$

namely

- u : solution of (1.1) $\Longrightarrow u_\lambda$: solution of (1.1)

Given $M > 0$, conseder a forward self-similar solution $U_M(t,x)$ such that

$$U_M(t,x) = \frac{1}{t}\Phi\Big(\frac{x}{\sqrt{t}}\Big), \quad \int_{\mathbb{R}^2} U_M(t,x)\,dx = M,$$

where

- $\Phi \geq 0, \quad \Phi \in L^1 \cap L^\infty$.

Existence and uniqueness of forward self-similar solutions

Biler, Applicationes Mathematicae (Warsaw), 1995
Biler-Karch-Laurençot-Nadzieja, Math. Meth. Appl. Sci., 2006
Naito-Suzuki, Taiwanese J. Math, 2004

1. Φ is radially symmetric.
2. Φ exists if and only if $0 < M < 8\pi$.
3. For each $0 < M < 8\pi$, the uniqueness of Φ up to the translation of the space variable holds.
4. For $0 < M < 8\pi$, let U_M be the radially symmetric with respect to the origin. Then

$$0 < U_M(t,x) \leq \frac{C}{t} e^{-|x|^2/t}, \quad t > 0, \; x \in \mathbb{R}^2.$$

Convergence to a forward self-similar solution

- Biler-Karch-Laurençot-Nadzieja, Math. Meth. Appl. Sci., 2006
  u: nonnegative radial solution to (KS)
  $M := \int_{\mathbb{R}^2} u_0(x)\,dx < 8\pi$.
  $$\hat{u}(t,r) := \int_{|x|<r} u(t,x)\,dx, \quad \hat{U}_M(t,r) := \int_{|x|<r} U_M(t,x)\,dx$$
  $$\lim_{t\to\infty} \|\hat{u}(t) - \hat{U}_M(t)\|_{L^\infty(0,\infty)} = 0$$
- u: nonnegative solution to (KS), $\quad M := \int_{\mathbb{R}^2} u_0(x)\,dx < 8\pi$
  $$\|u(t) - U_M(t)\|_{L^p} = o(t^{-1+1/p}) \text{ as } t \to \infty \quad (1 \le p \le \infty)$$
  Blanchet-Dolbeault-Perthame, Electron. J. Differential Equations, 2006 $\quad p = 1,\ u_0 \log u_0, |x|^2 u_0 \in L^1$
  N', Adv. Differential Equations, 2011 $\quad 1 \le p \le \infty,\ u_0 \in L^1$

1.2. The critical case $\int_{\mathbb{R}^2} u_0\,dx = 8\pi$ I

- Biler-Karch-Laurençot-Nadzieja, Math. Meth. Appl. Sci., 2006
  radial solutions
  - Global existence
  - Convergence to a stationary solution
  $$w_b(x) = \frac{8b}{(|x|^2+b)^2}, \ b > 0$$

  Stationary solutions:
  $$w_{b,x_0}(x) = \frac{8b}{(|x-x_0|^2+b)^2}, \quad b > 0, \ x_0 \in \mathbb{R}^2$$
  $$\int_{\mathbb{R}^2} w_{b,x_0}(x)\,dx = 8\pi$$

1.2. The critical case $\int_{\mathbb{R}^2} u_0\, dx = 8\pi$ II

- Blanchet-Carrillo-Masmoudi, Comm. Pure Appl. Math., 2008

  $u_0 \log u_0, |x|^2 u_0 \in L^1$.

  $\lim_{t\to\infty} u(t,x)dx = 8\pi\delta_{x_0}(x)$ in the sense of measure

  $$x_0 = \frac{1}{8\pi}\int_{\mathbb{R}^2} x u_0(x)\, dx : \text{ the center of mass of } u_0$$

- Senba, Adv. Differential Equations, 2009

  $\exists\, u_0 \geq 0$: radial $\int_{\mathbb{R}^2} u_0\, dx = 8\pi,\ |x|^2 u_0 \in L^1 \cap L^\infty$

  $$\lim_{t\to\infty} \frac{\|u(t)\|_{L^\infty}}{(\log t)^2} = \lim_{t\to\infty} \frac{u(t,0)}{(\log t)^2} = C > 0$$

1.2. The critical case $\int_{\mathbb{R}^2} u_0\, dx = 8\pi$ III

- Naito-Senba, preprint.
  Let $0 < b_1 < b_2 < \infty$.
  Then $\exists\, u_0 \geq 0$: radial, $\int_{\mathbb{R}^2} u_0\, dx = 8\pi$, $|x|^2 u_0 \notin L^1$ s.t.

$$w_{b_1}, w_{b_2} \in \omega(u_0),$$

$$w_b(x) = \frac{8b}{(|x|^2 + b)^2}, \quad b > 0 \quad \text{(stationary solution)}$$

$\omega(u_0)$: $\omega-$limit set of u_0 with respect to L^∞ topology

- For some choices of u_0, the solution goes to a stationary solution as $t \to \infty$.

In the critical case, the dynamics of (KS) is complicated.

2. Local exitence, uniqueness and regularity of mild solutions

$$
\text{(KS)} \quad \begin{cases} \partial_t u = \Delta u - \nabla \cdot (u(\nabla N * u)), & t > 0, \;\; x \in \mathbb{R}^2, \\ u|_{t=0} = u_0, & x \in \mathbb{R}^2. \end{cases}
$$

$$
N(x) = \frac{1}{2\pi} \log \frac{1}{|x-y|}, \quad \nabla N(x) = -\frac{1}{2\pi} \frac{x}{|x|^2}
$$

The equation in (KS) is very similar to the vorticity equation in $\mathbb{R}^2$:

$$\text{(VE)} \quad \begin{cases} \partial_t \omega = \Delta\omega - \nabla\cdot(\omega(\nabla^\perp N * \omega)), & t > 0,\ x \in \mathbb{R}^2, \\ \omega|_{t=0} = \omega_0, & x \in \mathbb{R}^2. \end{cases}$$

$$\nabla^\perp N(x) = -\frac{1}{2\pi}\frac{x^\perp}{|x|^2}, \quad x^\perp = (x_2, -x_1),\ x = (x_1, x_2)$$

- Giga, Miyakawa and Osada, Arch. Rational Mech. Anal., 96(1986)
- Kato, Differential Integral Equations, 7 (1994)
- Ben-Artzi, Arch. Rational Mech. Anal., 128 (1994)
- Brézis, Arch. Rational Mech. Anal., 128 (1994)

Definition 2.1 (mild solutions)

Let $0 < T < \infty$. *Given* $u_0 \in L^1$, *a function* $u : [0,T) \times \mathbb{R}^2 \to \mathbb{R}$ *is said to be a mild solution of* (KS) *on* $[0,T)$ *if*

1. $u \in C([0,T); L^1) \cap C((0,T); L^{4/3})$,
2. $\sup_{0<t<T} \left(t^{1/4} \|u(t)\|_{4/3} \right) < \infty$,
3. $u(t) = e^{t\Delta} u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta} (u(s)(\nabla N * u)(s))\, ds, \; 0 < t < T$,

$$(e^{t\Delta} f)(x) = \int_{\mathbb{R}^2} G(t, x-y) f(y)\, dy,$$

$$G(t,x) = \frac{1}{4\pi t} \exp(-\frac{|x|^2}{4t}).$$

A function $u : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ *is said to be a global mild solution of* (KS) *with initial data* u_0 *if* u *is a mild solution of* (KS) *on* $[0,T)$ *for any* $T \in (0,\infty)$.

Proposition 2.1 (Local existence, uniqueness and regularity)

Suppose $u_0 \in L^1$. *Then there exists* $T = T(u_0) \in (0, \infty)$ *such that the Cauchy problem* (KS) *has a unique mild solution* u *on* $[0, T)$. *Moreover,* u *satisfies the following properties:*

1. $u(t) \to u_0$ *in* L^1 *as* $t \to 0$.
2. *For every* $1 \leq q \leq \infty$, $u \in \dot{C}_{1-1/q,T}(L^q)$, *that is,*

$$\sup_{0<t<T} t^{1-1/q}\|u(t)\|_q < \infty, \quad \lim_{t \to 0} t^{1-1/q}\|u(t)\|_q = 0.$$

3. *For every* $\ell \in \mathbb{Z}_+, \alpha \in \mathbb{Z}_+^2$ *and* $1 < q < \infty$,

$$\sup_{0<t<T} t^{1-1/q+|\alpha|/2+\ell}\|\partial_t^\ell \partial_x^\alpha u(t)\|_q < \infty,$$

Proposition ctd.

4. For every $\ell \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^2$ and $2 - \min\{1, |\alpha|\} < q < \infty$,

$$\sup_{0<t<T} t^{1/2-1/q+|\alpha|/2+\ell} \|\partial_t^\ell \partial_x^\alpha (\nabla N * u)(t)\|_q < \infty,$$

5. u is a classical solution of $\partial_t u = \Delta u - \nabla \cdot (u(\nabla N * u))$ in $(0,T) \times \mathbb{R}^2$.
6. $\displaystyle\int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x) dx, \quad 0 < t < T.$
7. If $u_0 \geq 0$ but $u_0 \neq 0$, then $u(t,x) > 0$ for all $(t,x) \in (0,T) \times \mathbb{R}^2$.
8. If $u_0 \log(1 + |x|) \in L^1$, then $u(t) \log(1 + |x|) \in L^1, \quad 0 < t < T.$

3. Decreasing rearrangements

$f : \mathbb{R}^d \to \mathbb{R}$: measurable, $\theta \in \mathbb{R}$,

$$\{f > \theta\} := \{x \in \mathbb{R}^d : \ f(x) > \theta\},$$
$$|f > \theta| := |\{x \in \mathbb{R}^d : \ f(x) > \theta\}|,$$

where $|A|$ stands for the Lebesgue measure of a measurable set A. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function vanishing at infinity in the sense that

$$||f| > \theta| < \infty \qquad \text{for all } \ \theta > 0.$$

Definition 3.1 (Decreasing rearrangements)

The distribution function μ_f *of* f *is defined by*

$$\mu_f(\theta) := ||f| > \theta|, \qquad \theta \geq 0,$$

the decreasing rearrangement f^* *of* f *is defined through*

$$f^*(s) := \inf\{\theta \geq 0 : \ \mu_f(\theta) \leq s\}, \qquad s \geq 0$$

(it is a generalized inverse of μ_f*),*
the symmetric rearrangement, or Schwarz symmetrization of f*, denoted by* $f^\sharp : \mathbb{R}^d \to \mathbb{R}$*, is defined by*

$$f^\sharp(x) := f^*(c_d|x|^d),$$

where c_d *is the volume of the unit ball in* $\mathbb{R}^d$*.*

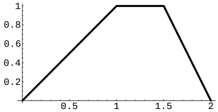


Figure 1: function f(x)

$$f(x) = \begin{cases} 0 & (x \leq 0,\ x \geq 2) \\ x & (0 < x < 1) \\ 1 & (1 \leq x \leq \frac{3}{2}) \\ 2(2-x) & (\frac{3}{2} < x < 2) \end{cases}$$

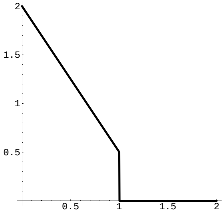


Figure 2: distribution function

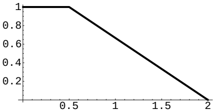


Figure 3: decreasing rearrangement

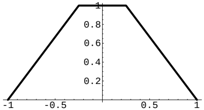


Figure 4: Schwarz symmetrization

Some basic properties about rearrangements are the following:

1. $||f| > \theta| = |f^\sharp > \theta| = |\{s \geq 0 |\, f^*(s) > \theta\}|, \;\; \theta > 0.$
2. f^* is non-increasing and right-continuous on $[0, \infty)$.
3. $f^*(0) = \|f\|_{L^\infty(\mathbb{R}^d)}, \quad f^*(\infty) = 0.$
4. If f is continuous and bounded on $\mathbb{R}^d$, then f^* and $f^\sharp$ are continuous and bounded on $[0, \infty)$ and $\mathbb{R}^d$, respectively.
5. $(f+g)^*(s_1+s_2) \leq f^*(s_1) + g^*(s_2)$ for all $s_1, s_2 > 0$.

Proposition 3.1

1. *For every Borel measurable function* $\Phi : \mathbb{R} \to [0, \infty)$,

$$\int_{\mathbb{R}^d} \Phi(|f(x)|)\, dx = \int_{\mathbb{R}^d} \Phi(f^\sharp(x))\, dx = \int_0^\infty \Phi(f^*(s))\, ds.$$

2. *Let* $f, g : \mathbb{R}^d \to \mathbb{R}$ *be integrable on* $\mathbb{R}^d$ *such that*

$$\int_0^s f^*(\sigma)\, d\sigma \leq \int_0^s g^*(\sigma)\, d\sigma \qquad \textit{for all } s > 0.$$

Then

$$\int_{\mathbb{R}^d} \Phi(|f(x)|)\, dx \leq \int_{\mathbb{R}^d} \Phi(|g(x)|)\, dx$$

for all convex functions $\Phi : [0, \infty) \to [0, \infty)$ *with* $\Phi(0) = 0$.

Proposition ctd.

3. (The Hardy-Littlewood inequality) Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then, for every $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |f||g|\, dx \leq \int_{\mathbb{R}^d} f^\sharp g^\sharp\, dx = \int_0^\infty f^* g^*\, ds.$$

4. (Contraction property) For every $p \in [1, \infty]$ and $f, g \in L^p(\mathbb{R}^d)$,

$$\|f^* - g^*\|_{L^p(0,\infty)} = \|f^\sharp - g^\sharp\|_{L^p(\mathbb{R}^d)} \leq \|f - g\|_{L^p(\mathbb{R}^d)}.$$

5. (The Pólya-Szegö inequality) For every $p \in [1, \infty]$ and $f \in W^{1,p}(\mathbb{R}^d)$, one has that $f^\sharp \in W^{1,p}(\mathbb{R}^d)$ and

$$\|\nabla f^\sharp\|_{L^p(\mathbb{R}^d)} \leq \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

For the properties of decreasing rearrangements, see the following, for example.

1. C. Bandle, Isoperimetric Inequalities and Applications, Pitman, London, 1980.
2. E. H. Lieb and M. Loss, Analysis, Graduate Studies in Mathematics, 14, Ameri. Math. Soc., Providence, RI, 2001.
3. J. Mossino, Inégalités Isopérimétriques et Applications en Physique, Hermann, Paris, 1984.
4. J.M. Rakotoson, Réarrangement Relatif: un instrument d'estimation dans les problèmes aux limites, Springer-Verlag, Berlin, 2008.

Lemma 3.1

$v : (0, T) \times \mathbb{R}^2 \to \mathbb{R}$ *smooth,* *radially symmetric in* x*, such that* $v(t) \in L^1 \cap L^\infty$ *for all* $t \in (0, T)$ *and*

$$\partial_t v = \Delta v - \nabla \cdot (v(\nabla N * v)) \quad \textit{in} \quad (0, T) \times \mathbb{R}^2.$$

Define $\varphi(t, s) := v(t, x), \quad s = \pi |x|^2, \quad \Phi(t, s) := \int_0^s \varphi(t, \sigma)\, d\sigma.$

Then

$$\int_{\mathbb{R}^2} v(t, x)\, dx = \int_0^\infty \varphi(t, s) ds, \qquad t \in [0, T), \tag{3.1}$$

$$\partial_t \varphi(t, s) = 4\pi \partial_s (s \partial_s \varphi(t, s)) + \partial_s \left(\varphi(t, s) \int_0^s \varphi(t, \sigma)\, d\sigma \right), \tag{3.2}$$

$$\partial_t \Phi(t, s) = 4\pi s \partial_s^2 \Phi(t, s) + \Phi(t, s) \partial_s \Phi(t, s). \tag{3.3}$$

Proof of Lemma 3.1

We observe that

$$\partial_t v = \Delta v - \nabla\cdot(v(\nabla N * v)) = \Delta v - \langle \nabla v, \nabla N * v\rangle - v\underbrace{\nabla\cdot(\nabla N * v)}_{-v}$$

$$= \Delta v - \langle \nabla v, \nabla N * v\rangle + v^2.$$

By $v(t,x) = \varphi(t,s),\ s = \pi|x|^2$, we have

$$\partial_t v - \Delta v = \partial_t\varphi - 4\pi\partial_s(s\partial_s\varphi).$$

Next, $-\langle \nabla v, \nabla N * v\rangle$ is rewritten as

$$-\langle \nabla v, \nabla N * v\rangle(t,x) = \partial_s\varphi(t,s)\int_{\mathbb{R}^2}\frac{\langle x, x-y\rangle}{|x-y|^2}\varphi(t,\pi|y|^2)\,dy. \tag{3.4}$$

Let $|x| \neq 0$. Put $y = Oz$, where O is an orthogonal matrix with $x = |x|Oe_1$, $e_1 = (1,0)$. Then

$$\int_{\mathbb{R}^2} \frac{\langle x, x-y\rangle}{|x-y|^2}\varphi(t,\pi|y|^2)\,dy = \int_{\mathbb{R}^2} \frac{|x|^2 - |x|\langle e_1, z\rangle}{||x|e_1 - z|^2}\varphi(t,\pi|z|^2)\,dz.$$

Introducing the polar coordinate $z_1 = r\cos\theta, z_2 = r\sin\theta$ gives

$$\begin{aligned}&\int_{\mathbb{R}^2} \frac{|x|^2 - |x|\langle e_1, z\rangle}{||x|e_1 - z|^2}\varphi(t,\pi|z|^2)\,dz \\ &= \int_0^\infty \varphi(t,\pi r^2)\left(\int_0^{2\pi} \frac{|x|^2 - |x|r\cos\theta}{|x|^2 - 2|x|r\cos\theta + r^2}\,d\theta\right) r dr.\end{aligned} \tag{3.5}$$

Putting $\tau = r/|x|$, we have

$$\begin{aligned}&\int_0^{2\pi} \frac{|x|^2 - |x|r\cos\theta}{|x|^2 - 2|x|r\cos\theta + r^2}\,d\theta = \int_0^{2\pi} \frac{1 - \tau\cos\theta}{1 - 2\tau\cos\theta + \tau^2}\,d\theta \\ &= \int_0^{2\pi} \frac{d\theta}{1 - \tau e^{i\theta}} = \begin{cases} 2\pi & (\tau < 1), \\ 0 & (\tau > 1). \end{cases} \qquad (i = \sqrt{-1})\end{aligned}$$

Then, by $\sigma = \pi r^2, s = \pi |x|^2$,

$$\int_{\mathbb{R}^2} \frac{|x|^2 - |x|\langle e_1, z\rangle}{||x|e_1 - z|^2} \varphi(t, \pi|z|^2)\, dz = 2\pi \int_0^{|x|} \varphi(t, \pi r^2) r dr$$
$$= \int_0^s \varphi(t, \sigma)\, d\sigma.$$

Therefore

$$-\langle \nabla v, \nabla N * v\rangle + v^2 = \partial_s \varphi(t, s) \left(\int_0^s \varphi(t, \sigma)\, d\sigma \right) + \varphi^2(t, s)$$
$$= \partial_s \left(\varphi(t, s) \int_0^s \varphi(t, \sigma)\, d\sigma \right).$$

Hence,

$$\partial_t \varphi(t,s) = 4\pi \partial_s (s \partial_s \varphi(t,s)) + \partial_s \left(\varphi(t,s) \int_0^s \varphi(t,\sigma)\, d\sigma \right).$$

Integrating this equation from 0 to s with respect to the variable s, we obtain

$$\partial_t \Phi(t,s) = 4\pi s \partial_s^2 \Phi(t,s) + \Phi(t,s) \partial_s \Phi(t,s).$$

For the nonnegative initial data $u_0 \in L^1$, let u be a nonnegative mild solution of (KS) in $[0, T)$ and let u^* denote its decreasing rearrangement with respect to x, and set

$$H(t,s) := \int_0^s u^*(t,\sigma)\, d\sigma, \quad 0 < t < T, \ s \geq 0.$$

- If u is radially symmetric in x and non-increasing in $|x|$, then

$$u(t,x) = u^*(t, \pi|x|^2), \quad 0 < t < T, \ x \in \mathbb{R}^2$$

and

$$\partial_t H - 4\pi s \partial_s^2 H - H \partial_s H = 0.$$

In the general case, we give the following propositions about the regularity and a differential equation of H.

Proposition 3.2

It hold that for every $p \in (1, \infty)$,

1. $H(t, 0) = 0$ *and* $H(t, \infty) = \int_{\mathbb{R}^2} u_0 \, dx$ *for all* $0 < t < T$,
2. $H \in BC([0, T) \times [0, \infty))$ *and* $H(0, s) = \int_0^s u_0^* \, d\sigma$ *for all* $s > 0$,
3. $\partial_s H \in BC((T_0, T) \times (0, \infty)) \cap L^\infty(0, T; L^1(0, \infty))$ *for all* $0 < T_0 < T$,
4. $\partial_s^2 H \in L^\infty(T_0, T; L^p(s_0, \infty))$ *for all* $0 < T_0 < T$ *and* $s_0 > 0$,
5. $\partial_t H \in L^\infty(T_0, T; L^p(0, R))$ *for all* $0 < T_0 < T$ *and* $R > 0$.

Proposition 3.3

It holds that for almost all $t \in (0, T)$,

$$\partial_t H - 4\pi s \partial_s^2 H - H \partial_s H \leq 0 \quad \text{a.a. } s > 0, \tag{3.6}$$

where

$$H(t,s) := \int_0^s u^*(t,\sigma)\, d\sigma, \quad 0 < t < T, \; s > 0.$$

To prove (5) of Proposition 3.2 and the differential inequality (3.3) in Proposition 3.3, we need to study the regularity of u^* with respect to the time variable t.

Proposition 3.4 (Comparison principle)

u : *a nonnegative mild solution of* (KS) *in* $[0, T)$ *with nonnegative initial data* $u_0 \in L^1$,
v : *a nonnegative radially symmetric mild solution to* (KS) *with nonnegative radially symmetric initial data* $v_0 \in L^1$. *Set*

$$v_0(x) := \varphi_0(\pi|x|^2), \quad v(t,x) := \varphi(t, \pi|x|^2).$$

If

$$\int_0^s u_0^*(\sigma)\, d\sigma \le \int_0^s \varphi_0(\sigma)\, d\sigma, \quad \forall\, s > 0,$$

then

$$\int_0^s u^*(t,\sigma)\, d\sigma \le \int_0^s \varphi(t,\sigma)\, d\sigma, \quad \forall\, 0 < t < T \;\; s > 0.$$

Proof of Proposition3.4

Put $H(t,s) = \int_0^s u^*(t,\sigma)\,ds, \quad \Phi(t,s) = \int_0^s \varphi(t,\sigma)\,ds$

1. For $0 < t < T, \ s > 0,$

$$\partial_t H - 4\pi s\partial_s^2 H - H\partial_s H \le 0, \quad \partial_t \Phi - 4\pi s\partial_s^2 \Phi - \Phi\partial_s \Phi = 0.$$

2. $H(t,0) = \Phi(t,0) = 0, \quad 0 < t < T.$
3. For $0 < t < T,$

$$H(t,\infty) = \int_0^\infty u^*(t,\sigma)\,d\sigma = \int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx$$
$$= \int_0^\infty u_0^*(\sigma)\,d\sigma.$$
$$\Phi(t,\infty) = \int_0^\infty \varphi_0(\sigma)\,d\sigma.$$

Hence $H(t,\infty) \le \Phi(t,\infty), \ 0 < t < T.$

4. $H(0,s) \le \Phi(0,s), \ s > 0.$

4. Subcritical case: Convergence to a forward self-similar solution

Given $M > 0$, consider a forward self-similar solution U_M of (KS) such that

$$U_M(t,x) = \frac{1}{t}\Phi\Big(\frac{x}{\sqrt{t}}\Big), \quad \int_{\mathbb{R}^2} U_M(t,x)\,dx = M,$$

where $\Phi \geq 0, \quad \Phi \in L^1 \cap L^\infty$.

Φ satisfies the following:

$$\nabla \cdot (\nabla\Phi - \Phi(\nabla N * \Phi)) + \Phi = 0 \ \text{ in } \ \mathbb{R}^2,$$

$$(\nabla N * \Phi)(x) := -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\Phi(y)\,dy.$$

Existence, uniqueness

Biler, Applicationes Mathematicae (Warsaw), 1995
Biler-Karch-Laurençot-Nadzieja, Math. Meth. Appl. Sci., 2006
Naito-Suzuki, Taiwanese J. Math, 2004

1. Φ is radially symmetric
2. Φ exists if and only if $0 < M < 8\pi$,
3. For each $0 < M < 8\pi$, the uniqueness of Φ up to the translation of the space variable holds.

Remarks (i) $\Phi(x) > 0$ $(x \in \mathbb{R}^2)$, $|x| \mapsto \Phi(x)$ is decreasing.

(ii) $0 < U_M(t,x) \leq \dfrac{C}{t} e^{-|x|^2/(4t)}$

In what follows, we discuss the following for the subcritical case:

$$M := \int_{\mathbb{R}^2} u_0 \, dx < 8\pi,$$

U_M : the forward self-similar solution with $\displaystyle\int_{\mathbb{R}^2} U_M(t,x)\, dx = M.$

- $u(t,\cdot) \to U_M(t,\cdot)$ in L^p $(t \to \infty)$ $(1 \le p \le \infty)$
- Convergence rates

4.1. Approach by entropy method

u: nonnegative solution to (KS)

Theorem 4.1

Blanchet-Dolbeault-Perthame, Electron. J. Differential Equations, **2006** *(2006)*

Assume $u_0 \log u_0, |x|^2 u_0 \in L^1(\mathbb{R}^2), \ M := \int_{\mathbb{R}^2} u_0(x)\,dx < 8\pi.$

Then

$$\lim_{t\to\infty} \|u(t) - U_M(t)\|_{L^1} = 0.$$

Their proof relies on

- rescaled transformations
- entropy method.

Free energy inequality

Free energy:

$$F[u] := \underbrace{\int_{\mathbb{R}^2} u \log u \, dx}_{\text{entropy}} - \underbrace{\frac{1}{2} \int_{\mathbb{R}^2} u\psi \, dx}_{\text{potential energy}},$$

$$\psi := N * u, \quad N(x) := \frac{1}{2\pi} \log \frac{1}{|x|}.$$

Lemma 4.1 (Free energy inequality)

For the nonnegative solution of (KS), it holds that

$$F[u(t)] + \int_0^t \int_{\mathbb{R}^2} u|\nabla \log u - \nabla\psi|^2 \, dxds \le F[u_0] \quad (t > 0).$$

Formal proof of the free energy inequality

$$\begin{aligned}\frac{d}{dt}\int u\log u\,dx &= \int(\partial_t u)\log u\,dx + \int \partial_t u\,dx \\ &= \int(\Delta u)\log u\,dx - \int\{\nabla\cdot(u\nabla\psi)\}\log u\,dx \\ &\quad + \underbrace{\int \nabla\cdot(\nabla u - u\nabla\psi)\,dx}_{=0} \\ &= -\int\frac{|\nabla u|^2}{u}\,dx + \int\langle\nabla u,\nabla\psi\rangle\,dx.\end{aligned}$$

Next

$$\frac{d}{dt}\int u\psi\,dx = \int(\partial_t u)\psi\,dx + \int u\partial_t\psi\,dx = 2\int(\partial_t u)\psi\,dx,$$

because, by $-\Delta\psi = u$,

$$\int u\partial_t\psi\,dx = -\int \Delta\psi\partial_t\psi\,dx = -\int \psi\partial_t\Delta\psi\,dx = \int \psi\partial_t u\,dx.$$

Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int u\psi\,dx &= \int(\partial_t u)\psi\,dx \\ &= \int(\Delta u)\psi\,dx - \int\{\nabla\cdot(u\nabla\psi)\}\psi\,dx \\ &= -\int\langle\nabla u, \nabla\psi\rangle\,dx + \int u|\nabla\psi|^2\,dx.\end{aligned}$$

Hence

$$\frac{d}{dt}\left(\int u\log u\,dx-\frac{1}{2}\int u\psi\,dx\right)$$
$$=-\int\left(\frac{|\nabla u|^2}{u}-2\langle\nabla u,\nabla\psi\rangle+u|\nabla\psi|^2\right)\,dx$$
$$=-\int\left(\left|\frac{\nabla u}{\sqrt{u}}\right|^2-2\langle\nabla u,\nabla\psi\rangle+|\sqrt{u}\nabla\psi|^2\right)\,dx$$
$$=-\int\left|\frac{\nabla u}{\sqrt{u}}-\sqrt{u}\nabla\psi\right|^2\,dx=-\int\left|\sqrt{u}\nabla\log u-\sqrt{u}\nabla\psi\right|^2\,dx$$
$$=-\int u|\nabla\log u-\nabla\psi|^2\,dx.$$

This implies

$$\frac{d}{dt}\left(\int u\log u\,dx-\frac{1}{2}\int u\psi\,dx\right)+\int u|\nabla\log u-\nabla\psi|^2\,dx=0.\quad\square$$

Outline of Proof of Theorem 4.1

Rescaled transformations

$$u(t,x) := \frac{1}{R^2(t)} v(\tau, y),$$

$$\tau = \log R(t), \quad y = \frac{x}{R(t)}, \quad R(t) := \sqrt{1+2t}$$

$$(\mathrm{KS})_R \begin{cases} \partial_\tau v = \Delta v - \nabla \cdot (v(\nabla \omega - y)), & \tau > 0, \ y \in \mathbb{R}^2, \\ \omega = \dfrac{1}{2\pi} \log \dfrac{1}{|y|} * v, & \tau > 0, \ y \in \mathbb{R}^2, \\ v(0,y) = u_0(y), & y \in \mathbb{R}^2. \end{cases}$$

self-similar solutions of (KS) $\iff$ stationary solutions of $(\mathrm{KS})_R$

$U_M(t,x)$ $\qquad$ $V_M(y)$

$$\lim_{t\to\infty} \|u(t) - U_M(t)\|_{L^1} = 0 \iff \lim_{\tau\to\infty} \|v(\tau) - V_M\|_{L^1} = 0$$

Entropy method

Rescaled free energy:

$$F^R[v] := \underbrace{\int_{\mathbb{R}^2} v \log v \, dy}_{\text{entropy}} - \underbrace{\frac{1}{2}\int_{\mathbb{R}^2} v\omega \, dy}_{\text{potential energy}} + \underbrace{\frac{1}{2}\int_{\mathbb{R}^2} |y|^2 v \, dy}_{\text{second moment}},$$

$$\omega := \frac{1}{2\pi} \log \frac{1}{|y|} * v$$

- (Free energy inequality for $F^R[v]$)

$$F^R[v(\tau)] + \int_0^\tau \int_{\mathbb{R}^2} v|\nabla \log v - (\nabla\omega - y)|^2 \, dyds \leq F^R[v_0]$$

$$\lim_{\tau\to\infty} F^R[v(\tau)] = F^R[V_M],$$

$$F^R[V_M] := \int_{\mathbb{R}^2} V_M \log V_M \, dy - \frac{1}{2}\int_{\mathbb{R}^2} V_M \Omega_M \, dy + \frac{1}{2}\int_{\mathbb{R}^2} |y|^2 V_M \, dy,$$

$$\Omega_M := \frac{1}{2\pi} \log \frac{1}{|y|} * V_M.$$

$$F^R[v(\tau)] - F^R[V_M] = \underbrace{\int_{\mathbb{R}^2} v(\tau) \log \frac{v(\tau)}{V_M} \, dy}_{\to 0} - \underbrace{\frac{1}{2}\int_{\mathbb{R}^2} |\nabla \omega(\tau) - \nabla \Omega_M|^2 \, dy}_{\to 0}$$

By the Csisz'ar-Kullback inequality

$$\|v(\tau) - V_M\|_{L^1}^2 \le 2M \underbrace{\int_{\mathbb{R}^2} v(\tau) \log \frac{v(\tau)}{V_M} \, dy}_{\text{relative entropy}} \to 0 \quad (\tau \to \infty)$$

Therefore,

$$\|u(t) - U_M(t)\|_{L^1} \to 0 \quad (t \to \infty).$$

4.2. Approach by rescaling method

Theorem 4.2

N', Adv. Differential Equations, 16 (2011)

Assumption : $u_0 \geq 0,\ u_0 \in L^1(\mathbb{R}^2),\ M := \int_{\mathbb{R}^2} u_0\, dx < 8\pi$

For $1 \leq p \leq \infty$,

$$\|u(t) - U_M(t)\|_{L^p} = o(t^{-1+1/p}) \quad \textit{as} \quad t \to \infty$$

Remarks

- The entropy method requires

$$u(t) \log u(t),\ |x|^2 u(t) \in L^1,\ t \geq 0.$$

- $u_0 \log u_0,\ |x|^2 u_0 \in L^1$ are not assumed in this theorem, so we need a different method from the entropy method to prove Theorem 4.2.

Outline of Proof of Theorem 4.2

The proof relies on the rescaling method:

$$\lim_{\lambda\to\infty} \|u_\lambda(1) - U_M(1)\|_{L^p} = 0$$

for $1 \le p \le \infty$, where

$$u_\lambda(t,x) := \lambda^2 u(\lambda^2 t, \lambda x)$$

- Put $\lambda = \sqrt{t}$. Then

$$t^{1-1/p}\|u(t)-U_M(t)\|_{L^p} = \|u_{\sqrt{t}}(1) - U_M(1)\|_{L^p} \to 0 \quad (t \to \infty)$$

Proposition 4.1

N', Integral Differential Equations 24 (2011)

$$1 \le p \le \infty, \ M := \int_{\mathbb{R}^2} u_0 \, dx < 8\pi.$$

1. $\|u(t)\|_{L^p} \le \|U_M(t)\|_{L^p}, \quad t > 0,$

   U_M *is the radially symmetric self-similar solution with*

   $$\int_{\mathbb{R}^2} U_M(t,x)\, dx = M$$

2. $\sup_{t>0} t^{1-1/p}\|u(t)\|_{L^p} \le C(M,p)$

Remark By $0 < U_M(t,x) \le \dfrac{C}{t} e^{-|x|^2/(4t)}$,

$$\|U_M(t)\|_{L^p} \le C(M,p) t^{-1+1/p}$$

Proposition 4.2

$1 \le p \le \infty,\ \ell \ge 0, n \ge 0.$

$$\sup_{t>0} t^{1-1/p+\ell/2+n} \|\partial_t^n \partial_x^\ell u(t)\|_{L^p} \le C(M, p, \ell, n)$$

Proof

$$u(t) = e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u(s)(\nabla N * u)(s))\, ds$$

$\forall\, \delta > 0,$

$$t^\delta u(t) = \delta \int_0^t e^{(t-s)\Delta}(s^{\delta-1}u(s))\, ds - \int_0^t \nabla \cdot e^{(t-s)\Delta}(s^\delta u(s)(\nabla N * u)(s))\, ds$$

By this expression of u, we derive Proposition 6.2 by induction on ℓ, n.

- $u_\lambda(t,x) := \lambda^2 u(\lambda^2 t, \lambda x)$ is the solution of (KS) with the initial data $u_{0,\lambda}(x) := \lambda^2 u_0(\lambda x)$.

  By $\int_{\mathbb{R}^2} u_{0,\lambda}(x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx = M$,

  for $1 \le p \le \infty,\ \ell \ge 0, n \ge 0,$

$$\sup_{t>0} t^{1-1/p+\ell/2+n} \|\partial_t^n \partial_x^\ell u_\lambda(t)\|_{L^p} \le C(M,p,\ell,n).$$

Remark The constants $C(M,p,\ell,n)$ are independent of λ

- For any $\{\lambda_j\}$ satisfying $\lambda_j \nearrow \infty$ $(j \nearrow \infty)$, there exist a subsequence of $\{\lambda_j\}$, denote it by $\{\lambda_j\}$ again, and $U \in C^\infty((0,\infty)\times\mathbb{R}^2)$ such that

$$\lim_{j\to\infty} \partial_t^n \partial_x^\ell u_{\lambda_j}(t,x) = \partial_t^n \partial_x^\ell U(t,x)$$

locally uniformly in $(0,\infty)\times\mathbb{R}^2$. $U \geq 0$

- $\displaystyle\int_{\mathbb{R}^2} u_{\lambda_j}(t,x)\,dx = M = \int_{\mathbb{R}^2} U(t,x)\,dx$
- $\displaystyle\lim_{j\to\infty} \|u_{\lambda_j}(t) - U(t)\|_{L^1} = 0, \quad t > 0.$
- By $\|\partial_x u_{\lambda_j}(t)\|_{L^p}, \|\partial_x U(t)\|_{L^p} \le C(M,p)t^{-1/2+1/p}$ $(1 \le \forall p \le \infty)$ and the Sobolev inequalities,

$$\lim_{j\to\infty} \|u_{\lambda_j}(t) - U(t)\|_{L^p} = 0, \quad \forall t > 0, \quad 1 < \forall p \le \infty$$

A crucial part of the proof is to show

- $U(t,x) = U_M(t,x)$

Once we get this relation, we conclude

$$\lim_{\lambda\to\infty} \|u_\lambda(t) - U_M(t)\|_{L^p} = 0, \quad \forall t > 0$$

To prove $U(t,x) = U_M(t,x)$, we use the following result.

Gallagher-Gallay-Lions(Math. Nachr., 278(2005))

$f, g : \mathbb{R}^d \to [0, +\infty)$: continuous, $|x|^d f, |x|^d g \in L^1(\mathbb{R}^d)$.

(i) g: radially symmetric, non-increasing with respect to $|x|$,

(ii) $\displaystyle\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{R}^d} g(x)\,dx,$

(iii) $\displaystyle\int_0^s f^*(\sigma)\,d\sigma \le \int_0^s g^*(\sigma)\,d\sigma, \quad \forall\, s > 0,$

(iv) $\displaystyle\int_{\mathbb{R}^d} |x|^d f(x)\,dx = \int_{\mathbb{R}^d} |x|^d g(x)\,dx.$

Then $f = g$.

f^* is the decresing rearrangement of f.

We apply this result as $f(x) = U(t,x),\ g(x) = U_M(t,x)\ (x \in \mathbb{R}^2)$.

Claim $\displaystyle\int_0^s U^*(t,\sigma)\,d\sigma \le \int_0^s U_M^*(t,\sigma)\,d\sigma, \quad \forall\, s>0$

$s \mapsto U^*(t,s)$: decreasing rearrangement of $x \mapsto U(t,x)$
$s \mapsto U_M^*(t,s)$: decreasing rearrangement of $x \mapsto U_M(t,x)$

Proof of Claim The proof of this claim relies on the following:

- N' (2011) $M := \int_{\mathbb{R}^2} u_0\,dx$. Let u be the nonnegative solution of (KS). Then

$$\int_0^s u^*(t,\sigma)\,d\sigma \le \int_0^s U_M^*(t,\sigma)\,d\sigma, \quad \forall\, s>0$$

Since u_{λ_j} is the nonnegative solution of (KS) with the initial data u_{0,λ_j} and $\int_{\mathbb{R}^2} u_{0,\lambda_j}\,dx = \int_{\mathbb{R}^2} u_0\,dx = M$, we also have

$$\int_0^s u_{\lambda_j}^*(t,\sigma)\,d\sigma \le \int_0^s U_M^*(t,\sigma)\,d\sigma, \quad \forall\, s>0$$

By $\|u_{\lambda_j}^*(t) - U^*(t)\|_{L^1(0,\infty)} \le \|u_{\lambda_j}(t) - U(t)\|_{L^1} \to 0 \ (j\to\infty)$, the claim is deduced.

Claim $\int_{\mathbb{R}^2} |x|^2 U(t,x)\,dx = \int_{\mathbb{R}^2} |x|^2 U_M(t,x)\,dx$

Proof of Claim We note that U and U_M are the solutions of the Cauchy problem (KS) with the initial data $M\delta_0$, where δ_0 is the Dirac $\delta-$function at the origin:

$$(\text{KS}) \begin{cases} \partial_t w = \Delta w - \nabla \cdot (w(\nabla N * w)), & t > 0,\ x \in \mathbb{R}^2, \\ w|_{t=0} = M\delta_0, & x \in \mathbb{R}^2 \end{cases}$$

By the second moment identity,

$$\int_{\mathbb{R}^2} |x|^2 w(t,x)\,dx = \underbrace{\int_{\mathbb{R}^2} |x|^2 M\delta_0(x)\,dx}_{=0} + 4\Big(1 - \frac{M}{8\pi}\Big)t$$

Hence the claim is deduced.

5. Dynamics of (KS) with critical mass 8π

In this section, we consider the case $\int_{\mathbb{R}^2} u_0\,dx = 8\pi$.
By the conservation of mass and the second moment identity,

$$\int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0\,dx = 8\pi, \quad t>0,$$
$$\int_{\mathbb{R}^2} |x|^2 u(t,x)\,dx = \int_{\mathbb{R}^2} |x|^2 u_0(x)\,dx + 4M\underbrace{\left(1-\frac{M}{8\pi}\right)}_{=0}t, \quad t>0$$

$$(M = \int_{\mathbb{R}^2} u_0\,dx)$$

- The second moment of u is conserved.
- The large-time behavior of u heavily depends on whether the second moment of u_0 is finite or not.

In the case where the second moment of u_0 is finite, Blanchet-Carrillo-Masmoudi proved the following.

Theorem 5.1

Let u_0 be in L^1 and nonnegative on $\mathbb{R}^2$ and $\int_{\mathbb{R}^2} u_0\, dx = 8\pi$. Suppose that

$$u_0 \log u_0,\ |x|^2 u_0 \in L^1.$$

Then there exists a nonnegative weak solution of $(\mathrm{KS})_\psi$ globally in time such that

$$\lim_{t\to\infty} u(t,x)dx = 8\pi\delta_{x_0}(x) \quad \textit{in the sense of measure,}$$

where δ_{x_0} is the Dirac distribution at x_0 and x_0 is the center of mass of u_0, namely

$$x_0 = \frac{1}{8\pi}\int_{\mathbb{R}^2} x u_0(x)\, dx.$$

Remark 5.1

- For their construction of the weak solution, assumption $u_0 \log u_0 \in L^1$ is required.
- Theorem 5.1 holds for the nonnegative mild solution u without $u_0 \log u_0 \in L^1$, because

$$u(t) \log u(t) \in L^1 \quad \text{for} \quad t > 0.$$

In fact, by Proposition 2.1,

$$u(t) \in L^p \ \ \text{for} \quad t > 0, \ 1 \leq p \leq \infty.$$

By this and

$$(1+u)\log(1+u) \leq C \times \begin{cases} u & (0 \leq u \leq 1), \\ u^2 & (u > 1) \end{cases},$$

we obtain

$$\int_{\mathbb{R}^2} (1 + u(t,x)) \log(1 + u(t,x)) \, dx < \infty.$$

Next, by the second moment identity,

$$\int_{\mathbb{R}^2} |x|^2 u(t,x) \, dx = \int_{\mathbb{R}^2} |x|^2 u_0(x) \, dx < \infty \quad \text{for} \quad t > 0.$$

From this and $u(t) \in L^1$,

$$\int_{\mathbb{R}^2} u(t,x) \log(1 + |x|) \, dx < \infty \quad \text{for} \quad t > 0.$$

Then Lemma 5.1 mentioned below ensures that

$$u(t) \log u(t) \in L^1 \quad \text{for} \quad t > 0.$$

Lemma 5.1

If a nonnegative function $f \in L^1$ satisfies

$$f \log(1 + |x|), (1 + f) \log(1 + f) \in L^1,$$

then

$$\begin{aligned}\int_{\mathbb{R}^2} f |\log f| \, dx \leq & \int_{\mathbb{R}^2} (1 + f) \log(1 + f) \, dx \\ & + 2\alpha \int_{\mathbb{R}^2} f \log(2 + |x|) \, dx \\ & + \frac{1}{e} \int_{\mathbb{R}^2} \frac{1}{(2 + |x|)^\alpha} \, dx,\end{aligned}$$

where $2 < \alpha < \infty$.

Proof of Lemma 5.1

We claim that for $a \geq 0, b > 0$,

$$a|\log a| \leq (1+a)\log(1+a) + 2a|\log b| + e^{-1}b. \qquad (5.1)$$

In fact, since $|(a/b)\log(a/b)| \leq e^{-1}$ for $a/b \leq 1$, we have

$$a|\log a| \leq e^{-1}b + a|\log b|.$$

By $|\log(a/b)| \leq |\log((a+1)/b)|$ for $a/b > 1$,

$$|\log a| \leq \log(1+a) + 2|\log b|.$$

Hence $a|\log a| \leq (1+a)\log(1+a) + 2a|\log b|$.
Thus we obtain (5.1).
Putting $a = f(x), b = (2+|x|)^{-\alpha}$ $(2 < \alpha < \infty)$ in (5.1) yields that

$$f(x)|\log f(x)| \leq (1+f(x))\log(1+f(x)) + 2\alpha f(x)\log(2+|x|) + e^{-1}(2+|x|)^{-\alpha}.$$

Integrating this inequality on $\mathbb{R}^2$ completes the proof.

We next consider large-time behavior in the case

$$\int_{\mathbb{R}^2} |x|^2 u_0(x)\, dx = \infty.$$

We recall that the stationary solutions

$$w_{b,x_0}(x) = \frac{8b}{(|x-x_0|^2+b)^2} \quad (x \in \mathbb{R}^2)$$

satisfy the following:

1. $\displaystyle\int_{\mathbb{R}^2} |x| w_{b,x_0}(x)\, dx < \infty, \quad \int_{\mathbb{R}^2} |x|^2 w_{b,x_0}(x)\, dx = \infty.$
2. $\displaystyle\int_{\mathbb{R}^2} w_{b,x_0}(x)\, dx = 8\pi, \quad \frac{1}{8\pi}\int_{\mathbb{R}^2} x w_{b,x_0}(x)\, dx = x_0.$

To study convergence to a stationary solution, Blanchet-Carlen-Carrillo, J. Funct. Anal., 262 (2012) introduced the following Lyapunov functional $\mathcal{H}_{b,x_0}$:

$$\mathcal{H}_{b,x_0}[f] = \int_{\mathbb{R}^2} \left(\sqrt{f(x)} - \sqrt{w_{b,x_0}(x)} \right)^2 w_{b,x_0}^{-1/2}(x)\, dx \qquad (5.2)$$

for $f \in L^1,\ f \geq 0$.
When x_0 is the origin, we denote w_{b,x_0} and $\mathcal{H}_{b,x_0}[f]$ by w_b and $\mathcal{H}_b[f]$, respectively, namely,

$$w_b(x) = \frac{8b}{(|x|^2 + b)^2} \quad (x \in \mathbb{R}^2),$$

$$\mathcal{H}_b[f] = \int_{\mathbb{R}^2} \left(\sqrt{f(x)} - \sqrt{w_b(x)} \right)^2 w_b^{-1/2}(x)\, dx.$$

Remark 5.2

If $\mathcal{H}_{b,x_0}[f] < \infty$ for $f \in L^1,\ f \geq 0$, then

$$\int_{\mathbb{R}^2} |x| f(x)\, dx < \infty,$$
$$\int_{\mathbb{R}^2} |x|^2 f(x)\, dx = \infty.$$

(See Lemma 5.2 mentioned below)

Theorem 5.2 (López Gómez-Nagai-Yamada)

Let $u_0 \in L^1$ *be a nonnegative initial data satisfying* $\int_{\mathbb{R}^2} u_0 \, dx = 8\pi$. *Assume that*

$$\mathcal{H}_b[u_0] < \infty \quad \textit{for some} \quad b > 0.$$

Then, the unique (nonnegative) mild solution u *of* (KS) *is globally defined in time and for any* $\tau > 0$ *there exists* $b_\tau > 0$ *such that for every* $1 \le p \le \infty$,

$$\|u(t)\|_p \le \|w_{b_\tau}\|_p \quad \textit{for all} \quad t \ge \tau. \tag{5.3}$$

If, in addition, $u_0 \in L^\infty$, *then there also exists* $b_0 > 0$ *such that for every* $1 \le p \le \infty$,

$$\|u(t)\|_p \le \|w_{b_0}\|_p \quad \textit{for all} \quad t \ge 0.$$

Theorem 5.3 (Löpez Gömez-Nagai-Yamada)

Let $u_0 \in L^1$ *be a nonnegative initial data satisfying* $\int_{\mathbb{R}^2} u_0 \, dx = 8\pi$, *and assume that*

$$\mathcal{H}_b[u_0] < \infty \quad \text{for some} \quad b > 0.$$

Then for the unique nonnegative mild solution u *of* (KS), *it holds that*

$$\lim_{t\to\infty} \|u(t) - w_{b,x_0}\|_p = 0 \quad \text{for all} \quad 1 \le p \le \infty,$$

where x_0 *is the center of mass of* u_0, *namely*

$$x_0 = \frac{1}{8\pi} \int_{\mathbb{R}^2} x u_0(x) \, dx.$$

subsect.5.4

Proof of Thm5.3

Such results as Theorems 5.2 and 5.3 were first proved by Blanchet-Carlen-Carrillo, J. Funct. Anal., 262 (2012).
They assumed

$$F[u_0] := \int_{\mathbb{R}^2} u_0(x) \log u_0(x)\, dx + \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u_0(x) u_0(y) \log|x-y|\, dxdy < \infty,$$

$$\mathcal{H}_b[u_0] < \infty \quad \text{for some} \quad b > 0,$$

and proved that

$$\sup_{t \geq \tau} \|u(t)\|_p < \infty \quad \text{for all} \quad \tau > 0 \text{ and } \quad 1 \leq p < \infty,$$

$$\lim_{t \to \infty} \|u(t) - w_{b,x_0}\|_1 = 0.$$

- To prove their results by Blanchet-Carlen-Carrillo, they used, for constructing the solution of (KS), an involved discrete variational scheme (called the JKO scheme), attributable to Jordan-Kinderlehrer-Otto, SIAM J. Math. Anal., 29 (1998).
- Our proofs in Löpez Gömez-N'-Yamada rely on an appropriate treatment of the functional $\mathcal{H}_b$ through some classical rearrangement techniques and energy methods. So, our methods are radically different from those used by Blanchet-Carlen-Carrillo.

Summary: The dynamics of (KS) with critical mass known so far I

$$L^1_{+\text{cri}} := \{f \in L^1 \,|\, f \geq 0 \text{ on } \mathbb{R}^2, \int_{\mathbb{R}^2} f\,dx = 8\pi\},$$
$$\mathcal{M}_2 := \{f \in L^1_{+\text{cri}} \,|\, \int_{\mathbb{R}^2} |x|^2 f(x)\,dx < \infty\},$$
$$\mathcal{H}_{\text{finite}} := \{f \in L^1_{+\text{cri}} \,|\, \mathcal{H}_b[f] < +\infty \text{ for some } b > 0\},$$
$$\mathcal{MH}_\infty := \{f \in L^1_{+\text{cri}} \,|\, f \notin \mathcal{M}_2,\ \mathcal{H}_b[f] = +\infty \text{ for all } b > 0\}.$$

Then

$$L^1_{+\text{cri}} = \mathcal{M}_2 \cup \mathcal{H}_{\text{finite}} \cup \mathcal{MH}_\infty.$$

Summary: The dynamics of (KS) with critical mass known so far II

1. If $u_0 \in \mathcal{M}_2$, then u converges to $8\pi\delta_{x_0}$ as $t \to \infty$, where x_0 is the center of mass of u_0.
(Blanche-Carrillo-Masmoudi)
2. If $u_0 \in \mathcal{H}_{\text{finite}}$, then u converges to a stationary solution w_{b,x_0} as $t \to \infty$.
(Blanchet-Carlen-Carrillo, Löpez Gömez-N'-Yamada)
3. There exists an initial data $u_0 \in \mathcal{MH}_\infty$ for which the omega limit set of u_0 with respect to L^∞-topology contains two different stationary solutions.
(Naito-Senba)

5.1. Some properties of the entropy functional $\mathcal{H}_{b,x_0}$

- For $b > 0,\ x_0 \in \mathbb{R}^2$,

$$w_{b,x_0}(x) = \frac{8b}{(|x-x_0|^2+b)^2} \quad \text{(stationary solutions)},$$

$$\mathcal{H}_{b,x_0}[f] = \int_{\mathbb{R}^2} \left(\sqrt{f(x)} - \sqrt{w_{b,x_0}(x)}\right)^2 w_{b,x_0}^{-1/2}(x)\,dx.$$

- When $x_0 = 0$,

$$w_b(x) := w_{b,x_0}(x) = \frac{8b}{(|x|^2+b)^2},$$

$$\mathcal{H}_b[f] := \mathcal{H}_{b,x_0}[f] = \int_{\mathbb{R}^2} \left(\sqrt{f(x)} - \sqrt{w_b(x)}\right)^2 w_b^{-1/2}(x)\,dx.$$

Lemma 5.2

Suppose $b > 0$, $x_0 \in \mathbb{R}^2$ *and* $f \in L^1$ *satisfies* $f \geq 0$. *Then,*

1. $\mathcal{H}_{b,x_0}[w_{b,x_0}] = 0$ *and* $\mathcal{H}_{b,x_0}[w_{a,x_0}] = \infty$ *for all* $a > 0$, $a \neq b$,
2. $\mathcal{H}_{b,x_0}[f] < \infty$ *implies* $\mathcal{H}_{b,x_1}[f] < \infty$ *for all* $x_1 \in \mathbb{R}^2$,
3. $\mathcal{H}_{b,x_0}[f] < \infty$ *implies* $\mathcal{H}_{a,x_0}[f] = \infty$ *for all* $a > 0$, $a \neq b$,
4. $\mathcal{H}_{b,x_0}[f] < \infty$ *implies*

$$\int_{\mathbb{R}^2} \sqrt{b + |x|^2} f(x)\, dx \leq 16\pi b^{1/2} + (8b)^{1/4} \left(\|f\|_1^{1/2} + \|w_b\|_1^{1/2} \right) \sqrt{\mathcal{H}_b[f]}$$

   and, in particular, $|x| f \in L^1$.
5. $\mathcal{H}_{b,x_0}[f] < \infty$ *implies* $|x|^2 f \notin L^1$.

▸ Proof of Lemma 5.2

Theorem 5.4 (the entropy-entropy dissipation inequality)

Let u_0 *be such that*

$$u_0 \geq 0 \text{ on } \mathbb{R}^2, \qquad u_0 \in L^1, \qquad \int_{\mathbb{R}^2} u_0 = 8\pi, \tag{5.4}$$

and $\mathcal{H}_b[u_0] < \infty$ *for some* $b > 0$. *Then the mild solution* u *of* (KS) *in* $[0, T)$ *satisfies*

$$\mathcal{H}_b[u(t)] + \int_0^t \mathcal{D}[u(s)]\, ds \leq \mathcal{H}_b[u_0] \quad \text{for all } \; 0 < t < T, \tag{5.5}$$

where $\mathcal{D}[u]$ *is defined by*

$$\mathcal{D}[u] := 8 \int_{\mathbb{R}^2} |\nabla u^{1/4}|^2\, dx - \int_{\mathbb{R}^2} u^{3/2}\, dx. \tag{5.6}$$

▸ Proof of EEDI

We give a remark about the entropy dissipation $\mathcal{D}[u]$:

Lemma 5.3

Suppose $f \in L^1$, $f \geq 0$, $\int_{\mathbb{R}^2} f = 8\pi$ *and* $\nabla f^{1/4} \in L^2$. *Then*

$$\mathcal{D}[f] := 8 \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2 \, dx - \int_{\mathbb{R}^2} f^{3/2} \, dx \geq 0.$$

Moreover, $\mathcal{D}[f] = 0$ *if and only if* $f = w_{b,x_0}$ *for* $\exists\, b > 0, x_0 \in \mathbb{R}^2$.

Lemma 5.3 follows by applying the next lemma to the function $g := f^{1/4}$.

Lemma 5.4 (Del Pino-Dolbeault, J. Math. Pures Appl., 81(2002))

Suppose $g \in L^4$ *and* $|\nabla g| \in L^2$. *Then,*

$$\pi \int_{\mathbb{R}^2} |g|^6 \, dx \leq \int_{\mathbb{R}^2} |\nabla g|^2 \, dx \int_{\mathbb{R}^2} |g|^4 \, dx.$$

Moreover, the equality occurs if and only if $g = w_{b,x_0}^{1/4}$ *for* $\exists\, b > 0,\ x_0 \in \mathbb{R}^2$.

subsect.5.2

Proof of Lemma 5.2

(1) For $a, b > 0$ with $a \neq b$ and sufficiently large $|x|$, there exists a constant $C > 0$ such that

$$\left(\sqrt{w_{a,x_0}(x)} - \sqrt{w_{b,x_0}(x)}\right)^2 w_{b,x_0}^{-1/2}(x) \geq \frac{C}{|x|^2}$$

and, hence, $\mathcal{H}_{b,x_0}[w_{a,x_0}] = \infty$. By definition, $\mathcal{H}_{b,x_0}[w_{b,x_0}] = 0$.
(2) Property (2) follows easily from the fact that

$$\lim_{|x|\uparrow\infty} \frac{\left(\sqrt{f(x)} - \sqrt{w_{b,x_0}(x)}\right)^2 w_{b,x_0}^{-1/2}(x)}{\left(\sqrt{f(x)} - \sqrt{w_{b,x_1}(x)}\right)^2 w_{b,x_1}^{-1/2}(x)} = 1.$$

(3) To prove (3), let a, $b > 0$ with $a \neq b$. Then, it follows from

$$(z-x)^2 + (z-y)^2 \geq \frac{1}{2}(x-y)^2, \qquad x, y, z \in \mathbb{R},$$

that

$$\left(\sqrt{f}-\sqrt{w_{a,x_0}}\right)^2 w_{a,x_0}^{-1/2} \geq \frac{1}{2}\left(\sqrt{w_{b,x_0}}-\sqrt{w_{a,x_0}}\right)^2 w_{a,x_0}^{-1/2} - \left(\sqrt{f}-\sqrt{w_{b,x_0}}\right)^2 w_{a,x_0}^{-1/2}$$

in $\mathbb{R}^2$. Moreover, there exists a constant $C > 0$ such that

$$\left(\sqrt{f}-\sqrt{w_{b,x_0}}\right)^2 w_{a,x_0}^{-1/2} \leq C\left(\sqrt{f}-\sqrt{w_{b,x_0}}\right)^2 w_{b,x_0}^{-1/2}.$$

Therefore, integrating these estimates in $\mathbb{R}^2$, yields to

$$\mathcal{H}_{a,x_0}[f] \geq \frac{1}{2}\mathcal{H}_{a,x_0}[w_{b,x_0}] - C\mathcal{H}_{b,x_0}[f].$$

As, owing to (1), $\mathcal{H}_{a,x_0}[w_{b,x_0}] = \infty$, we find from this estimate that $\mathcal{H}_{a,x_0}[f] = \infty$, which concludes the proof of Part (3).

(4) Our proof of the estimate of Part (4) is based on the proof of Lemma 1.10 of Blanchet-Carlen-Carrillo. By the sake of completeness, we will give complete details here. We have

$$\int_{\mathbb{R}^2} \sqrt{b+|x|^2} f(x)\,dx = \underbrace{\int_{\mathbb{R}^2} \sqrt{b+|x|^2}\, w_b(x)\,dx}_{=I_1} + \underbrace{\int_{\mathbb{R}^2} \sqrt{b+|x|^2}\,(f(x)-w_b(x))\,dx}_{=I_2}.$$

By changing to polar coordinates, it is easily seen that

$$I_1 = \int_{\mathbb{R}^2} \frac{16b}{(b+|x|^2)^{3/2}}\,dx = 16\pi\sqrt{b}.$$

Moreover, as

$$\sqrt{b+|x|^2} = (8b)^{1/4} w_b^{-1/4}(x),$$

we have that

$$
\begin{aligned}
|I_2| &\leq \int_{\mathbb{R}^2} \sqrt{b+|x|^2}\,|f(x)-w_b(x)|\,dx \\
&= (8b)^{1/4} \int_{\mathbb{R}^2} \left|\sqrt{f(x)}+\sqrt{w_b(x)}\right| \left|\sqrt{f(x)}-\sqrt{w_b(x)}\right| w_b^{-1/4}(x)\,dx \\
&\leq (8b)^{1/4} \left(\int_{\mathbb{R}^2} \left(\sqrt{f}+\sqrt{w_b}\right)^2 dx \right)^{1/2} \\
&\qquad \times \left(\int_{\mathbb{R}^2} \left(\sqrt{f}-\sqrt{w_b}\right)^2 w_b^{-1/2}\,dx \right)^{1/2} \\
&= (8b)^{1/4} \|\sqrt{f}+\sqrt{w_b}\|_2 \sqrt{\mathcal{H}_b[f]} \\
&\leq (8b)^{1/4} \left(\|\sqrt{f}\|_2 + \|\sqrt{w_b}\|_2 \right) \sqrt{\mathcal{H}_b[f]} \\
&= (8b)^{1/4} \left(\|f\|_1^{1/2} + \|w_b\|_1^{1/2} \right) \sqrt{\mathcal{H}_b[f]}.
\end{aligned}
$$

Adding these estimates provides us with the estimate of Part (4), which implies $|x|f \in L^1$.

(5) It follows from the definition of w_b that

$$\begin{aligned} |x|^2 f(x) &= \sqrt{8b}\, w_b^{-1/2}(x) f(x) - bf(x) \\ &\geq \sqrt{8b}\, w_b^{-1/2}(x) \left[\frac{1}{2} w_b(x) - \left(\sqrt{f(x)} - \sqrt{w_b(x)}\right)^2\right] - bf(x) \\ &= \sqrt{2b}\, w_b^{1/2}(x) - \sqrt{8b} \left(\sqrt{f(x)} - \sqrt{w_b(x)}\right)^2 w_b^{-1/2}(x) - bf(x). \end{aligned}$$

Consequently, integrating in $\mathbb{R}^2$ shows that

$$\int_{\mathbb{R}^2} |x|^2 f(x)\, dx \geq \sqrt{2b} \int_{\mathbb{R}^2} \sqrt{w_b}\, dx - \sqrt{8b}\, \mathcal{H}_b[f] - b \int_{\mathbb{R}^2} f\, dx.$$

Therefore, $\int_{\mathbb{R}^2} |x|^2 f(x)\, dx = \infty$, because

$$\int_{\mathbb{R}^2} \sqrt{w_b}\, dx = \infty, \quad \mathcal{H}_b[f] < \infty, \quad \int_{\mathbb{R}^2} f\, dx < \infty.$$

For a rigorous proof, see Blanchet-Carlen-Carrillo and Julian-N'-Yamada.

Formal proof of Theorem 5.4

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_b[u(t)] &= \frac{d}{dt}\int_{\mathbb{R}^2}(\sqrt{u}-\sqrt{w_b})^2 w_b^{-1/2}\,dx \\ &= \int_{\mathbb{R}^2}\partial_t u(w_b^{-1/2}-u^{-1/2})\,dx \\ &= \int_{\mathbb{R}^2}\partial_t u\,w_b^{-1/2}\,dx-\int_{\mathbb{R}^2}\partial_t u\,u^{-1/2}\,dx.\end{aligned}$$

$$\int_{\mathbb{R}^2} \partial_t u(t) w_b^{-1/2}\,dx = (8b)^{-1/2} \int_{\mathbb{R}^2} \partial_t u(t)(|x|^2 + b)\,dx$$
$$= (8b)^{-1/2} \int_{\mathbb{R}^2} \Delta u(t)(|x|^2 + b)\,dx$$
$$\quad - (8b)^{-1/2} \int_{\mathbb{R}^2} \nabla \cdot (u(t)(\nabla N * u)(t))(|x|^2 + b)\,dx$$
$$= (8b)^{-1/2} \int_{\mathbb{R}^2} u(t) \underbrace{\Delta |x|^2}_{=4}\,dx + 2(8b)^{-1/2} \int_{\mathbb{R}^2} u(t) \langle x, (\nabla N * u)(t)\rangle\,dx.$$

Hence,

$$\int_{\mathbb{R}^2} \partial_t u(t) w_b^{-1/2}\,dx$$
$$= 4(8b)^{-1/2} \int_{\mathbb{R}^2} u(t)\,dx$$
$$\quad - 2(8b)^{-1/2} \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} u(t,x)u(t,y) \frac{\langle x, x-y\rangle}{|x-y|^2}\,dydx.$$

Replacing x and y of the integrand $u(t,x)u(t,y)\dfrac{\langle x, x-y\rangle}{|x-y|^2}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,x)u(t,y)\frac{\langle x, x-y\rangle}{|x-y|^2}\,dydx\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,y)u(t,x)\frac{\langle y, y-x\rangle}{|x-y|^2}\,dxdy,\end{aligned}$$

and hence,

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,x)u(t,y)\frac{\langle x, x-y\rangle}{|x-y|^2}\,dydx\\ &= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,x)u(t,y)\left(\frac{\langle x, x-y\rangle}{|x-y|^2}+\frac{\langle y, y-x\rangle}{|x-y|^2}\right)dydx\\ &= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(t,x)u(t,y)\,dydx\\ &= \frac{1}{2}\left(\int_{\mathbb{R}^2} u(t,x)\,dx\right)^2.\end{aligned}$$

Therefore, since $\int_{\mathbb{R}^2} u(t)\,dx = 8\pi$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} \partial_t u(t) w_b^{-1/2}\,dx &= 4(8b)^{-1/2}\int_{\mathbb{R}^2} u(t)\,dx \\ &\qquad - (8b)^{-1/2}\frac{1}{2\pi}\Big(\int_{\mathbb{R}^2} u(t,x)\,dx\Big)^2 \\ &= 0.\end{aligned}$$

Next,

$$\int_{\mathbb{R}^2} \partial_t u u^{-1/2}\,dx = \int_{\mathbb{R}^2} \Delta u u^{-1/2}\,dx - \int_{\mathbb{R}^2} \nabla\cdot(u(\nabla N * u))u^{-1/2}\,dx.$$

Then

$$\int_{\mathbb{R}^2} \Delta u u^{-1/2}\,dx = \frac{1}{2}\int_{\mathbb{R}^2} u^{-3/2}|\nabla u|^2\,dx = 8\int_{\mathbb{R}^2} |\nabla u^{1/4}|^2\,dx,$$

$$-\int_{\mathbb{R}^2} \nabla\cdot(u(\nabla N * u))u^{-1/2}\,dx = -\frac{1}{2}\int_{\mathbb{R}^2} u^{-1/2}\langle\nabla u, \nabla N * u\rangle\,dx$$

$$= -\int_{\mathbb{R}^2} \langle\nabla u^{1/2}, \nabla N * u\rangle\,dx = \int_{\mathbb{R}^2} u^{1/2}\underbrace{\nabla\cdot(\nabla N * u)}_{=-u}\,dx$$

$$= -\int_{\mathbb{R}^2} u^{3/2}\,dx.$$

Hence

$$\int_{\mathbb{R}^2} \partial_t u u^{-1/2}\, dx = 8 \int_{\mathbb{R}^2} |\nabla u^{1/4}|^2\, dx - \int_{\mathbb{R}^2} u^{3/2}\, dx = \mathcal{D}[u(t)].$$

Therefore

$$\frac{d}{dt}\mathcal{H}_b[u(t)] = \underbrace{\int_{\mathbb{R}^2} \partial_t u\, w_b^{-1/2}\, dx}_{=0} - \underbrace{\int_{\mathbb{R}^2} \partial_t u\, u^{-1/2}\, dx}_{=\mathcal{D}[u(t)]}$$
$$= -\mathcal{D}[u(t)],$$

from which the entropy-entropy dissipation inequality/equality (5.5) follows.

EEDI

5.2. Boundedness of the solutions

In this subsection, we will prove Theorem 5.2 after some lemmas and a theorem. ▸ thm5.2

As $f^\sharp = f$ if f is radially symmetric and non-increasing in $|x|$, we observe that

$$w_b(x) = w_b^\sharp(x) = w_b^*(\pi|x|^2), \quad x \in \mathbb{R}^2.$$

Here

$$w_b(x) = \frac{8b}{(|x|^2 + b)^2}, \qquad x \in \mathbb{R}^2$$

is the stationary solution of (KS), and, therefore, the decreasing rearrangement of $w_b(x)$ is given by

$$w_b^*(s) = \frac{8\pi^2 b}{(s + \pi b)^2}, \qquad s \geq 0. \tag{5.7}$$

Consequently,

$$\int_0^s w_b^* \, d\sigma = \frac{8\pi s}{s + \pi b}, \qquad s \geq 0. \tag{5.8}$$

Naturally, this implies $\int_0^\infty w_b^* \, d\sigma = 8\pi$ and

$$\int_s^\infty w_b^* \, d\sigma = 8\pi - \frac{8\pi s}{s + \pi b} = \frac{8\pi^2 b}{s + \pi b}$$

and hence,

$$\lim_{s \to \infty} \left(s \int_s^\infty w_b^* \, d\sigma \right) = 8\pi^2 b.$$

Lemma 5.5

Suppose f *satisfies*

$$f \geq 0 \quad \text{in } \mathbb{R}^2, \qquad f \in L^1, \qquad \int_{\mathbb{R}^2} f \, dx = 8\pi, \tag{5.9}$$

and

$$\liminf_{s \to \infty} \left(s \int_s^\infty f^*(\sigma) \, d\sigma \right) > 0. \tag{5.10}$$

Then there exist $b_0 > 0$ *and* $s_0 > 0$ *such that*

$$\int_0^s f^* \, d\sigma < \int_0^s w_{b_0}^* \, d\sigma \quad \text{for all } \ s \geq s_0.$$

If, in addition, $f \in L^\infty$*, then there exists* $b_1 \in (0, b_0)$ *such that*

$$\int_0^s f^* \, d\sigma < \int_0^s w_{b_1}^* \, d\sigma \quad \text{for all } \ s > 0.$$

Proof of Lemma 5.5

According to (5.10), there exist $b_0 > 0$ and $s_0 > 0$ such that

$$s \int_s^\infty f^* \, d\sigma > 8\pi^2 b_0 \quad \text{for all} \quad s \geq s_0,$$

which implies

$$\int_s^\infty f^* \, d\sigma > \frac{8\pi^2 b_0}{s + \pi b_0} \quad \text{for all} \quad s \geq s_0.$$

On the other hand, owing to Proposition 3.1, it follows from (5.9) that

$$\int_0^\infty f^* \, d\sigma = \int_{\mathbb{R}^2} f \, dx = 8\pi.$$

Thus, using (5.8), it becomes apparent that for all $s \geq s_0$,

$$\int_0^s f^* \, d\sigma = 8\pi - \int_s^\infty f^* \, d\sigma < 8\pi - \frac{8\pi^2 b_0}{s + \pi b_0} = \frac{8\pi s}{s + \pi b_0} = \int_0^s w_{b_0}^* \, d\sigma.$$

Subsequently, besides (5.10) and (5.9), we assume that $f \in L^\infty$. Naturally, for every $b_1 \in (0, b_0)$, we also have that for all $s \geq s_0$,

$$\int_0^s f^* \, d\sigma < \int_0^s w_{b_0}^* \, d\sigma = \frac{8\pi s}{s + \pi b_0} < \frac{8\pi s}{s + \pi b_1} = \int_0^s w_{b_1}^* \, d\sigma.$$

Let $b_1 < b_0$ be such that

$$0 < f^*(0) = \|f\|_{L^\infty(\mathbb{R}^2)} < 8/b_1.$$

Then there exists $\delta > 0$ such that

$$\int_0^s f^* \, d\sigma < \int_0^s w_{b_1}^* \, d\sigma = \frac{8\pi s}{s + \pi b_1} \qquad \text{for all } \; s \in [0, \delta].$$

This completes the proof if $\delta \geq s_0$, but, in general, $\delta < s_0$. So, suppose $\delta < s_0$. We should shorten b_1, if necessary, so that

$$\int_0^s f^* \, d\sigma < \int_0^s w_{b_1}^* \, d\sigma = \frac{8\pi s}{s + \pi b_1} \qquad \text{for all } \; s \in [\delta, s_0]. \quad (5.11)$$

Thanks to (5.10),

$$\int_0^s f^* \, d\sigma < \int_0^\infty f^* \, d\sigma = 8\pi \quad \text{for all} \quad s > 0.$$

On the other hand, we have that

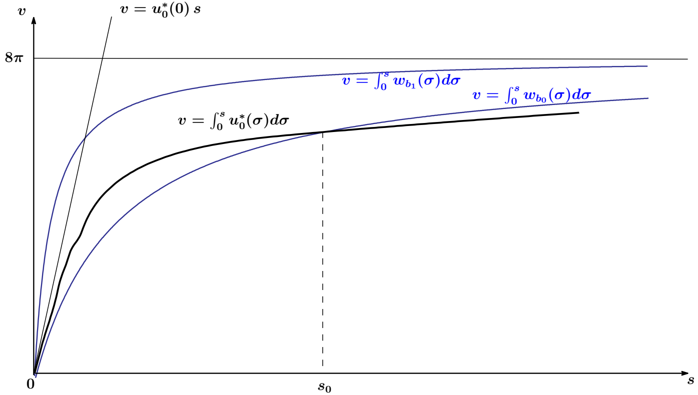
$$\lim_{b_1 \downarrow 0} \frac{8\pi s}{s + \pi b_1} = 8\pi \qquad \text{uniformly in} \quad [\delta, s_0].$$

Consequently, b_1 can be shortened, if necessary, to get (5.11). This ends the proof.

$$w_b(x) = \frac{8b}{(b+|x|^2)^2}, \quad w_b^*(s) = \frac{8\pi^2 b}{(\pi b + s)^2}, \quad \int_0^s w_b^*(\sigma)\, d\sigma = \frac{8\pi s}{\pi b + s}$$

$$\exists\, b_0 > 0,\ s_0 > 0 \text{ s.t. } \int_0^s u_0^*(\sigma)\, d\sigma \le \int_0^s w_{b_0}^*(\sigma)\, d\sigma, \quad s \ge s_0$$

$$\int_0^s u_0^*(\sigma)\, d\sigma \le u_0^*(0)\, s, \quad s \ge 0$$



Theorem 5.5

Let $u_0 \in L^1 \cap L^\infty$ be such that $u_0 \geq 0$, $\int_{\mathbb{R}^2} u_0 \, dx = 8\pi$, and

$$\liminf_{s \to \infty} \left(s \int_s^\infty u_0^* \, d\sigma \right) > 0. \tag{5.12}$$

Then the (unique) nonnegative mild solution u of (KS) is globally defined in time, and there exists $b > 0$ such that, for every $t > 0$, $s > 0$, and $p \in [1, \infty]$,

$$\int_0^s u^*(\sigma, t) \, d\sigma \leq \int_0^s w_b^*(\sigma) \, d\sigma \quad \text{and} \quad \|u(t)\|_p \leq \|w_b\|_p. \tag{5.13}$$

Proof of Theorem 5.5

According to Lemma 5.5, there exists $b > 0$ such that

$$\int_0^s u_0^* \, d\sigma < \int_0^s w_b^* \, d\sigma \quad \text{for all} \quad s > 0.$$

Define

$$H(t,s) = \int_0^s u^*(t,\sigma)\, d\sigma, \quad W(s) = \int_0^s w_b^*(\sigma)\, d\sigma.$$

Then

1. For $t > 0, s > 0$,

$$\partial_t H \leq 4\pi s \partial_s^2 H + H\partial_s H, \quad 4\pi s \partial_s^2 W + W\partial_s W = 0.$$

2. For $t > 0$,

$$H(t,0) = W(0) = 0, \quad H(t,\infty) = W(\infty) = 8\pi.$$

3. For $s > 0$, $H(0,s) < W(s)$.

Hence, by the comparison principle (Proposition 3.4),

$$H(t,s) \leq W(s), \quad t > 0, \; s \geq 0,$$

that is,

$$\int_0^s u^*(\sigma, t)\, d\sigma \leq \int_0^s w_b^*(\sigma)\, d\sigma, \quad t > 0, \; s \geq 0.$$

Taking $\Phi(u) = u^p \; (u \geq 0)$ for $1 < p < \infty$ in Proposition 3.1 (ii), we have

$$\int_{\mathbb{R}^2} u^p(t,x)\, dx \leq \int_{\mathbb{R}^2} w_b^p(x)\, dx.$$

Hence, this shows the global existence of unique mild solution u, and for every $1 < p < \infty$,

$$\|u(t)\|_p \leq \|w_b\|_p, \quad t > 0.$$

Letting $p \to \infty$ in this inequality, we obtain

$$\|u(t)\|_\infty \leq \|w_b\|_\infty, \quad t > 0.$$

Thus the proof of Theorem 5.5 is complete.

To prove Theorem 5.2 , we need the following lemma.

Lemma 5.6

Suppose f satisfies the following:

1. $f \geq 0$ *in* $\mathbb{R}^2$, $f \in L^1$, $\int_{\mathbb{R}^2} f\,dx = 8\pi$,
2. $\mathcal{H}_b[f] < \infty$ *for some* $b > 0$.

Then

$$\liminf_{s\to\infty} \left(s \int_s^\infty f^*\,d\sigma \right) \geq 2\pi^2 b. \tag{5.14}$$

In particular, (5.10) *is satisfied.*

Proof of Lemma 5.6

Setting

$$g := \sqrt{f} - \sqrt{w_b},$$

it is apparent that

$$f = w_b + h, \quad h := 2g\sqrt{w_b} + g^2. \tag{5.15}$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^2} g^2(x)(b + |x|^2)\,dx &= \sqrt{8b}\int_{\mathbb{R}^2} g^2(x) w_b^{-1/2}(x)\,dx \\ &= \sqrt{8b}\,\mathcal{H}_b[f] < \infty.\end{aligned} \tag{5.16}$$

For every $R > 1$, we have that

$$\int_{|x|\geq R} g^2(x)\,dx \leq R^{-2}\int_{|x|\geq R} |x|^2 g^2(x)\,dx,$$

and, hence, by (5.16),

$$\int_{|x|\geq R} g^2(x)\,dx = o\Big(R^{-2}\Big) \qquad \text{as } \ R \to \infty.$$

Similarly, since

$$\int_{|x|\geq R} \frac{w_b(x)}{|x|^2}\,dx = \int_{|x|\geq R} \frac{8b}{(b+|x|^2)^2|x|^2}\,dx \leq 4\pi b R^{-4},$$

it follows from Hölder's inequality that

$$\int_{|x|\geq R} \sqrt{w_b(x)}|g(x)|\,dx = \int_{|x|\geq R} \frac{\sqrt{w_b(x)}}{|x|}|g(x)||x|\,dx$$

$$\leq \left(\int_{|x|\geq R} \frac{w_b(x)}{|x|^2}\,dx\right)^{1/2} \Big(\int_{|x|\geq R} |g(x)|^2|x|^2\,dx\Big)^{1/2}$$

$$\leq 2\sqrt{\pi b}R^{-2}\Big(\int_{|x|\geq R} |g(x)|^2|x|^2\,dx\Big)^{1/2}$$

and, consequently, (5.16) implies

$$\int_{|x|\geq R} \sqrt{w_b(x)}|g(x)|\,dx = o\Big(R^{-2}\Big) \quad \text{as } \; R \to \infty.$$

Therefore, we find from (5.15) that

$$\int_{|x|\geq R} |h(x)|\,dx = o\Big(R^{-2}\Big) \quad \text{as } \; R \to \infty \tag{5.17}$$

As $w_b = f + (-h)$ and $(-h)^* = h^*$, applying the basic properties on rearrangements in Section 3, it is apparent that

$$w_b^*(2s) \leq f^*(s) + h^*(s) \qquad \text{for all } \; s > 0$$

and hence,

$$f^*(s) \geq w_b^*(2s) - h^*(s) \quad \text{for all } \; s > 0 \tag{5.18}$$

We will derive (5.14) from (5.18). To do it, we need to estimate

$$\int_s^\infty w_b^*(2\sigma)\, d\sigma \quad \text{and} \quad \int_s^\infty h^*(\sigma)\, d\sigma.$$

By (5.7), we find that

$$\int_s^\infty w_b^*(2\sigma)\, d\sigma = \frac{4\pi^2 b}{2s + \pi b}$$

and, hence,

$$\lim_{s\to\infty}\left(s\int_s^\infty w_b^*(2\sigma)\, d\sigma\right) = 2\pi^2 b \tag{5.19}$$

To conclude the proof of the lemma, it suffices to show that

$$\int_s^\infty h^*(\sigma)\, d\sigma \le \int_{|x|\ge(s/\pi)^{1/2}} |h(x)|\, dx \tag{5.20}$$

Indeed, suppose (5.20) holds. Then, by (5.17) we deduce that

$$s\int_s^\infty h^*(\sigma)\,d\sigma \le s\int_{|x|\ge(s/\pi)^{1/2}} |h(x)|\,dx \to 0 \quad \text{as} \quad s\to\infty \tag{5.21}$$

Therefore, combining (5.18), (5.19) and (5.21),

$$\begin{aligned}
&\liminf_{s\to\infty}\left(s\int_s^\infty f^*(\sigma)\,d\sigma\right)\\
&\ge \lim_{s\to\infty}\left(s\int_s^\infty w_b^*(2\sigma)\,d\sigma\right) - \lim_{s\to\infty}\left(s\int_s^\infty h^*(\sigma)\,d\sigma\right)\\
&= 2\pi^2 b
\end{aligned}$$

The proof of (5.20) can be accomplished as follows. Thanks to the Hardy-Littlewood inequality, for every $R>0$, we have that

$$\begin{aligned}
\int_{|x|<R} |h(x)|\,dx &= \int_{\mathbb{R}^2} |h(x)|\chi_{B_R}(x)\,dx\\
&\le \int_{\mathbb{R}^2} h^\sharp(x)\chi_{B_R}^\sharp(x)\,dx = \int_{|x|<R} h^\sharp(x)\,dx,
\end{aligned}$$

where χ_{B_R} stands for the characteristic function of the ball $B_R := B_R(0)$, and we have used that $\chi_{B_R}^\sharp = \chi_{B_R}$. As, due to Proposition 3.1(i),

$$\int_{\mathbb{R}^2} |h|\,dx = \int_{\mathbb{R}^2} h^\sharp\,dx,$$

we infer from the previous estimate that

$$\begin{aligned}\int_{|x|\geq R} |h(x)|\,dx &= \int_{\mathbb{R}^2} |h(x)|\,dx - \int_{|x|<R} |h(x)|\,dx \\ &\geq \int_{\mathbb{R}^2} h^\sharp(x)\,dx - \int_{|x|<R} h^\sharp(x)\,dx \\ &= \int_{|x|\geq R} h^\sharp(x)\,dx.\end{aligned}$$

Therefore, by the definition of $h^\sharp$,

$$\begin{aligned}\int_{|x|\geq R} |h(x)|\,dx &\geq \int_{|x|\geq R} h^\sharp(x)\,dx = \int_{|x|\geq R} h^*(\pi|x|^2)\,dx \\ &= 2\pi\int_R^\infty h^*(\pi\rho^2)\rho\,d\rho = \int_{\pi R^2}^\infty h^*(\sigma)\,d\sigma.\end{aligned}$$

Taking $s = \pi R^2$ in this inequality shows (5.20):

$$\int_s^\infty h^*(\sigma)\,d\sigma \leq \int_{|x|\geq (s/\pi)^{1/2}} |h(x)|\,dx.$$

Thus the proof of Lemma 5.6 is complete.

Proof of Theorem 5.2

Let $T_{max} > 0$ denote the maximal existence time of the unique mild solution of (KS). By Proposition 2.1,

$$u(t) \in L^1 \cap L^\infty \quad \text{for all} \quad t \in (0, T_{max}).$$

Moreover, by Lemma 5.3, we have that

$$\mathcal{D}(u(t)) \geq 0 \quad \text{for all} \quad t \in (0, T_{max}).$$

Thus, owing to Theorem 5.4, we have that

$$\mathcal{H}_b[u(t)] \leq \mathcal{H}_b[u_0] < \infty \quad \text{for all} \quad t \in (0, T_{max}). \tag{5.22}$$

Consequently, it follows from Lemma 5.6 that

$$\liminf_{s\to\infty} \left(s \int_s^\infty u^*(\tau, \sigma)\, d\sigma \right) \geq 2\pi^2 b.$$

As the function $t \mapsto u(t+\tau)$ is a mild solution of (KS) in $[0, T_{max} - \tau)$ with nonnegative initial data $u(\tau) \in L^1 \cap L^\infty$, according to Theorem 5.5 $u(t+\tau)$ must be globally defined in time and (5.3) holds:

$$\exists\, b_\tau > 0 \quad s.t. \quad \sup_{t \geq \tau} \|u(t)\|_p \leq \|w_{b_\tau}\|_p \quad \text{for all} \quad 1 \leq p \leq \infty.$$

In particular, $T_{max} = \infty$ and the proof is complete.

5.4. Convergence to a stationary solution

This section proves Theorem 5.3. thm5.3
Thus, throughout it, we will assume that the initial data $u_0 \in L^1$ satisfy

$$u_0 \geq 0, \quad \int_{\mathbb{R}^2} u_0 \, dx = 8\pi \quad \text{and} \quad \mathcal{H}_b[u_0] < \infty \quad \text{for some} \quad b > 0.$$

By Theorem 5.2, we already know that the unique mild solution u of (KS) is nonnegative and globally defined in time. Moreover,

$$\sup_{t \geq 1} \|u(t)\|_p < \infty \quad \text{for all} \quad 1 \leq p \leq \infty. \tag{5.23}$$

The proof of Theorem 5.3 will follow after some lemmas of technical nature.

Lemma 5.7

The following estimates hold:

$$\sup_{t\geq 2} \|\nabla u(t)\|_p < \infty \quad (2 \leq \forall\, p < \infty),$$

$$\sup_{t\geq 2} \int_t^{t+1} (\|\partial_t u(s)\|_2^2 + \|\Delta u(s)\|_2^2)\, ds < \infty.$$

Lemma 5.8

For every $t > 0$ and $R > 1$ the following uniform integrability estimate holds:

$$\int_{|x|>R} (b+|x|^2)^{1/2} u(t,x)\,dx \\ \leq \int_{|x|>R} (b+|x|^2)^{1/2} w_b(x)\,dx + \Phi(b,R) \left(\Psi(b) + \| |x| w_b \|_1^{1/2} \right), \tag{5.24}$$

where

$$\Phi(b,R) := (8b)^{1/4} \mathcal{H}_b[u_0]^{1/2} R^{-1/2},$$

$$\Psi(b) := \left(16\pi b^{1/2} + 2(8b)^{1/4} (8\pi)^{1/2} \sqrt{\mathcal{H}_b[u_0]} \right)^{1/2}.$$

Proof of Lemma 5.8

A direct calculation shows that

$$(b+|x|^2)^{1/2}u = (b+|x|^2)^{1/2}w_b+(8b)^{1/4}w_b^{-1/4}(\sqrt{u}-\sqrt{w_b})(\sqrt{u}+\sqrt{w_b}),$$

where $u = u(t,x)$ and $w_b = w_b(x)$. Thus, integrating this identity on $|x| > R$, we have that

$$\int_{|x|>R}(b+|x|^2)^{1/2}u(t,x)\,dx \le \int_{|x|>R}(b+|x|^2)^{1/2}w_b(x)\,dx+(8b)^{1/4}I,$$

where

$$I := \int_{|x|>R} w_b^{-1/4}(x)\left(\sqrt{u(t,x)}-\sqrt{w_b(x)}\right)\left(\sqrt{u(t,x)}+\sqrt{w_b(x)}\right)\,dx.$$

Using Hölder's inequality and

$$\mathcal{H}_b[u(t)] \leq \mathcal{H}_b[u_0] \quad (t > 0) \qquad \text{(by (5.22))}$$

and setting $\Omega := \{|x| > R\}$, we can estimate I as follows.

$$\begin{aligned} I &\leq \left(\int_{|x|>R} w_b^{-1/2} (\sqrt{u} - \sqrt{w_b})^2 \, dx \right)^{1/2} \left(\int_{|x|>R} (\sqrt{u} + \sqrt{w_b})^2 \, dx \right)^{1/2} \\ &\leq \mathcal{H}_b[u(t)] \|\sqrt{u} + \sqrt{w_b}\|_{L^2(\Omega)} \\ &\leq \mathcal{H}_b[u(t)] \left(\|\sqrt{u}\|_{L^2(\Omega)} + \|\sqrt{w_b}\|_{L^2(\Omega)} \right) \end{aligned}$$

$$\begin{aligned} \|\sqrt{u}\|_{L^2(\Omega)} &= \left(\int_{|x|>R} |x|^{-1} \cdot |x| u(t,x) \, dx \right)^{1/2} \\ &\leq R^{-1/2} \left(\int_{|x|>R} |x| u(t,x) \, dx \right)^{1/2}. \end{aligned}$$

Similarly,

$$\|\sqrt{w_b}\|_{L^2(\Omega)} \leq R^{-1/2} \left(\int_{|x|>R} |x| w_b(x)\, dx \right)^{1/2}.$$

Hence

$$I \leq \mathcal{H}_b[u_0] R^{-1/2} \times \left[\left(\int_{|x|>R} |x| u(t,x)\, dx \right)^{1/2} + \left(\int_{|x|>R} |x| w_b(x)\, dx \right)^{1/2} \right].$$

On the other hand, applying Lemma 5.2(iv) to $u(t)$, using the conservation of mass of u and (5.22), we get

$$\begin{aligned}\int_{\mathbb{R}^2} |x|u(t,x)\,dx &\le 16\pi b^{1/2} + (8b)^{1/4}\left(\|u(t)\|_1^{1/2} + \|w_b\|_1^{1/2}\right)\sqrt{\mathcal{H}_b[u(t)]} \\ &\le 16\pi b^{1/2} + (8b)^{1/4}\left(\|u_0\|_1^{1/2} + \|w_b\|_1^{1/2}\right)\sqrt{\mathcal{H}_b[u_0]} \\ &\le 16\pi b^{1/2} + 2(8b)^{1/4}(8\pi)^{1/2}\sqrt{\mathcal{H}_b[u_0]}\end{aligned}$$

and, therefore,

$$I \le \mathcal{H}_b[u_0]^{1/2} R^{-1/2}\left(\Psi(b) + \||x|w_b\|_1^{1/2}\right).$$

This concludes the proof.

The next result establishes the averaged large-time asymptotic of the solution.

Lemma 5.9

For every $1 \leq p \leq 2$*,*

$$\lim_{T\to\infty} \int_T^{T+1} \int_{\mathbb{R}^2} |u(t,x) - w_{b,x_0}(x)|^p \, dxdt = 0, \tag{5.25}$$

where x_0 *is the center of mass of* u_0*.*

Proof of Lemma 5.9

By the conservation of the center of mass

$$\frac{1}{8\pi}\int_{\mathbb{R}^2} xu(t,x)\,dx = \frac{1}{8\pi}\int_{\mathbb{R}^2} xu_0(x)\,dx = x_0$$

and the translational invariance of the problem in the space coordinate, we may assume $x_0 = 0$ without lost of generality. Let $\{t_n\}_{n\geq 1}$ be an arbitrary sequence of times such that

$$\lim_{n\to\infty} t_n = \infty$$

and consider the translated solutions

$$u_n(t,x) := u(t+t_n,x), \qquad 0 \leq t \leq 1,\ x \in \mathbb{R}^2.$$

Then

$$\sup_{n\geq 1}\sup_{0\leq t\leq 1}\|u_n(t)\|_{H^1}<\infty, \tag{5.26}$$

$$\sup_{n\geq 1}\int_0^1\|\partial_t u_n(t)\|_2^2\,dt<\infty. \tag{5.27}$$

By the proof of Lemma 5.8, we already know that

$$\sup_{n\geq 1}\sup_{0\leq t\leq 1}\int_{\mathbb{R}^2}|x|u_n(t,x)\,dx\leq \Psi^2(b)<\infty. \tag{5.28}$$

Now, we will show that for each $0\leq t\leq 1$,

$$\{u_n(t)\}_{n=1}^{\infty}\ \text{ is relatively compact in }\ L^2(\mathbb{R}^2). \tag{5.29}$$

Take any $t\in[0,1]$ and fix it. By (5.26),

$$\{u_n(t)\}_{n\geq 1}\ \text{ is bounded in }\ H^1.$$

Thus, by that fact that

embedding $H^1(B_R) \hookrightarrow L^2(B_R)$ compact for every $R > 0$,

we can extract a subsequence of $\{u_n(t)\}_{n\geq 1}$, relabeled by $\{u_n(t)\}_{n\geq 1}$, and a function $v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \|u_n(t) - v\|_{L^2(B_R)} = 0 \quad \text{for all} \quad R > 0. \tag{5.30}$$

We claim that, actually, $v \in L^2(\mathbb{R}^2)$ and that, along some subsequence,

$$\lim_{n\to\infty} \|u_n(t) - v\|_{L^2(\mathbb{R}^2)} = 0. \tag{5.31}$$

Indeed, by the convergence of $\{u_n(t)\}_{n\geq 1}$ to v in $L^2(B_R)$ for all $R > 0$, we can extract a subsequence, again labeled by n, such that

$$\lim_{n\to\infty} u_n(t,x) = v(x) \qquad a.e. \text{ in } \mathbb{R}^2.$$

As $\{u_n(t)\}_{n\geq 1}$ is bounded in $L^p(\mathbb{R}^2)$ for all $1 \leq p \leq \infty$, we also have

$$v \in L^p(\mathbb{R}^2) \quad \text{for all} \quad 1 \leq p \leq \infty.$$

Due to (5.28),

$$\sup_{n\geq 1} \int_{\mathbb{R}^2} |x| u_n(t,x)\, dx \leq \Psi^2(b) < \infty,$$

and hence, thanks to Fatou's lemma, we find that

$$\int_{\mathbb{R}^2} |x| v(x)\, dx \leq \Psi^2(b) < \infty.$$

Thus,

$$\sup_{n\geq 1} \int_{\mathbb{R}^2} |x| |u_n(t,x) - v(x)|\, dx \leq 2\Psi^2(b) < \infty. \tag{5.32}$$

Then, owing to (5.32), we find that for $R > 0$,

$$\int_{\mathbb{R}^2} |u_n(t) - v|^2\, dx = \int_{|x|<R} |u_n(t) - v|^2\, dx + \int_{|x|>R} |u_n(t) - v|^2\, dx$$

$$\leq \|u_n(t) - v\|^2_{L^2(B_R)} + R^{-1} \int_{|x|>R} |x||u_n(t) - v|^2\, dx$$

$$\leq \|u_n(t) - v\|^2_{L^2(B_R)} + CR^{-1} \int_{|x|>R} |x||u_n(t) - v|\, dx$$

$$\leq \|u_n(t) - v\|^2_{L^2(B_R)} + C\Psi^2(b)R^{-1}$$

for some nonnegative constant C. By this,

$$\limsup_{n\to\infty} \|u_n(t) - v\|_2^2 \leq 4C\Psi^2(b)R^{-1},$$

and then, by letting $R \to \infty$,

$$\lim_{n\to\infty} \|u_n(t) - v\|_2 = 0.$$

and hence, (5.29) holds.

We claim that, actually, $v \in L^2(\mathbb{R}^2)$ and that, along some subsequence, Next, owing to (5.27), we obtain that, for any $0 \leq t_1 < t_2 \leq 1$,

$$\|u_n(t_2) - u_n(t_1)\|_2 \leq \int_{t_1}^{t_2} \|\partial_t u_n(t)\|_2 \, dt$$
$$\leq |t_2 - t_1|^{1/2} \sup_{n \geq 1} \int_0^1 \|\partial_t u_n(t)\|_2^2 \, dt$$

and, therefore,

$\{u_n\}_{n\geq 1}$ is uniformly equicontinuos in $C([0,1]; L^2)$.

Then, by the Ascoli-Arzela theorem (see, e.g., Lemma 1 of Simon),

$\{u_n\}_{n\geq 1}$ is relatively compact in $C([0,1]; L^2)$.

Therefore, there exists $w \in C([0,1];L^2)$ and, along some subsequence, relabeled by n, we must have

$$\lim_{n\to\infty} u_n = w \quad \text{in} \quad C([0,1];L^2). \tag{5.33}$$

From (5.28) it follows that

$$\sup_{0\le t\le 1} \int_{\mathbb{R}^2} |x| w(t,x)\, dx \le \Psi^2(b) < \infty,$$

and from (5.33) it is easily seen that

$$\lim_{n\to\infty} u_n = w \quad \text{in} \quad C([0,1];L^1). \tag{5.34}$$

According to Theorem 5.4,

$$\mathcal{H}_b[u_n(t)] + \int_0^t \mathcal{D}[u_n(s)]\, ds \le \mathcal{H}_b[u_0], \qquad 0 \le t \le 1, \quad n \ge 1.$$

- For $f \in L^1, f \geq 0, \int_{\mathbb{R}^2} f\,dx = 8\pi, \nabla f \in L^1$,

$$\mathcal{D}[f] := 8 \int_{\mathbb{R}^2} |\nabla f^{1/4}|^2\,dx - \int_{\mathbb{R}^2} f^{3/2}\,dx \geq 0.$$

$\mathcal{D}[f] = 0 \Longleftrightarrow f = w_{b,x_0}$ for some $b > 0, x_0 \in \mathbb{R}^2$.

Thus,

$$\begin{aligned} 8 \int_0^1 \int_{\mathbb{R}^2} |\nabla u_n^{1/4}|^2\,dxdt &= \int_0^1 \mathcal{D}[u_n(t)]\,dt + \int_0^1 \|u_n(t)\|_{3/2}^{3/2}\,dt \\ &\leq \mathcal{H}_b[u_0] + \sup_{t \geq 1} \|u(t)\|_{3/2}^{3/2}. \end{aligned} \tag{5.35}$$

By (5.34),

$$\lim_{n \to \infty} u_n^{1/4} = w^{1/4} \qquad \text{in} \quad C([0,1]; L^4).$$

Thus, due to (5.35), we may assume that

$$\lim_{n\to\infty} \nabla u_n^{1/4} = \nabla w^{1/4} \quad \text{weakly in} \quad L^2((0,1)\times\mathbb{R}^2).$$

Hence,

$$\int_0^1 \int_{\mathbb{R}^2} |\nabla w^{1/4}|^2 \, dxdt \le \liminf_{n\to\infty} \int_0^1 \int_{\mathbb{R}^2} |\nabla u_n^{1/4}|^2 \, dxdt$$

and, therefore,

$$\int_0^1 \mathcal{D}[w(t)] \, dt \le \liminf_{n\to\infty} \int_0^1 \mathcal{D}[u_n(t)] \, dt. \tag{5.36}$$

Once again by Theorem 5.4, we also find that

$$\int_0^\infty \mathcal{D}[u(t)] \, dt \le \mathcal{H}_b[u_0].$$

Consequently, since

$$\int_0^1 \mathcal{D}[u_n(t)]\,dt = \int_0^1 \mathcal{D}[u(t+t_n)]\,dt = \int_{t_n}^{t_n+1} \mathcal{D}[u(s)]\,ds$$

for all $n \geq 1$, it becomes apparent that

$$\lim_{n\to\infty} \int_0^1 \mathcal{D}[u_n(t)]\,dt = 0.$$

Therefore, (5.36) entails

$$\int_0^1 \mathcal{D}[w(t)]\,dt = 0. \tag{5.37}$$

As, according to Lemma 5.3, we have $\mathcal{D}[w(t)] \geq 0$, the identity (5.37) implies

$$\mathcal{D}[w(t)] = 0 \quad \text{for all} \quad t \in [0,1] \setminus N,$$

where N is a subset of $[0,1]$ of measure zero. Consequently, once again by Lemma 5.3, for every $t \in [0,1] \setminus N$, there exist $b(t) > 0$ and $x_0(t) \in \mathbb{R}^2$ such that

$$w(t,x) = w_{b(t),x_0(t)}(x) = \frac{8b(t)}{(|x - x_0(t)|^2 + b(t))^2} \quad \text{on} \quad \mathbb{R}^2.$$

In what follows, we will show $\underline{x_0(t) = 0 \text{ and } b(t) = b.}$

By (5.24), we observe that

$$\sup_{n\geq 1}\int_{|x|>R}|x|u_n(t,x)\,dx\to 0 \quad\text{as}\quad R\to\infty.$$

Hence, since $u_n(t)\to w(t)$ in L^1 as $n\to\infty$, we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}xu_n(t,x)\,dx=\int_{\mathbb{R}^2}xw(t,x)\,dx=\int_{\mathbb{R}^2}xw_{b(t),x_0(t)}(x)\,dx$$
$$=8\pi x_0(t).$$

As we are assuming that the center of mass of u_0 is zero, by the conservation of the center of mass for $u(t)$, we have that

$$\int_{\mathbb{R}^2}xu_n(t,x)\,dx=\int_{\mathbb{R}^2}xu_0(x)\,dx=0.$$

Therefore, $\underline{x_0(t)=0}$ and, hence, for every $t\in[0,1]\setminus N$,

$$w(t,x)=w_{b(t)}(x)=\frac{8b(t)}{(|x|^2+b(t))^2},\quad\text{on}\quad\mathbb{R}^2.$$

By (5.34), for every $t \in [0,1] \setminus N$, there exists a subsequence $\{u_{n_j}(t)\}_{j\geq 1}$ of $\{u_n(t)\}_{n\geq 1}$ such that

$$\lim_{j\to\infty} u_{n_j}(t,x) = w_{b(t)}(x) \quad \text{a.e.} \quad \text{in } \mathbb{R}^2.$$

Then, thanks to Fatou's lemma, (5.22) implies that

$$\begin{aligned}
\mathcal{H}_b[w_{b(t)}] &= \int_{\mathbb{R}^2} (\sqrt{w_{b(t)}} - \sqrt{w_b})^2 w_b^{-1/2}\, dx \\
&\leq \liminf_{j\to\infty} \int_{\mathbb{R}^2} \left(\sqrt{u_{n_j}(t)} - \sqrt{w_b}\right)^2 w_b^{-1/2}\, dx \\
&= \liminf_{j\to\infty} \mathcal{H}_b[u_{n_j}(t)] = \liminf_{j\to\infty} \mathcal{H}_b[u(t+t_{n_j})] \leq \mathcal{H}_b[u_0].
\end{aligned}$$

Therefore,

$$\mathcal{H}_b[w_{b(t)}] \leq \mathcal{H}_b[u_0] < \infty.$$

Consequently, according to Lemma 5.2(i),

$$b(t) = b \quad \text{for all} \quad t \in [0,1] \setminus N$$

and, therefore,

$$w(t) = w_b \quad \text{for all} \quad t \in [0,1] \setminus N.$$

Since $w : [0,1] \to L^1 \cap L^2$ is continuous, we have

$$w(t) = w_b \quad \text{for all} \quad t \in [0,1].$$

Owing to (5.33) and (5.34), we also find that, for every $p = 1, 2$,

$$\begin{aligned} &\lim_{n\to\infty} \int_{t_n}^{t_n+1} \int_{\mathbb{R}^2} |u(t,x) - w_b(x)|^p \, dxdt \\ &= \lim_{n\to\infty} \int_0^1 \int_{\mathbb{R}^2} |u_n(t,x) - w_b(x)|^p \, dxdt \\ &= 0. \end{aligned}$$

This provides us with (5.25) for $p = 1, 2$.
The general case when $1 \le p \le 2$ follows from the following interpolation inequality: for every $1 \le q < p < r \le \infty$ and $\lambda \in [0, 1]$ with $1/p = \lambda/q + (1 - \lambda)/r$,

$$\|f\|_p \le \|f\|_q^{\lambda} \|f\|_r^{1-\lambda} \quad \text{for all} \quad f \in L^q \cap L^p.$$

Actually, for $1 < p < 2$,

$$\begin{aligned}
&\int_{t_n}^{t_n+1} \|u(t) - w_b\|_p^p \, dt \\
&\le \int_{t_n}^{t_n+1} \|u(t) - w_b\|_1^{(2-p)/p} \|u(t) - w_b\|_2^{(2p-2)/p} \, dt \\
&\le \Big(\int_{t_n}^{t_n+1} \|u(t) - w_b\|_1 \, dt \Big)^{(2-p)/p} \Big(\int_{t_n}^{t_n+1} \|u(t) - w_b\|_2 \, dt \Big)^{(2p-2)/p}
\end{aligned}$$

This ends the proof.

Proof of Theorem 5.3

thm5.3

As in Lemma 5.9, we may assume that the center of mass of u_0 is zero, that is, $x_0 = 0$. Take any sequence of times $\{t_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} t_n = \infty.$$

Due to Lemma 5.9, we have that

$$\lim_{n\to\infty} \int_{t_n}^{t_n+1} \|u(t) - w_b\|_2^2 \, dt = 0. \tag{5.38}$$

Thus, for every $n \geq 1$, there exists $s_n \in [t_n, t_n + 1]$ such that

$$\lim_{n\to\infty} u(s_n) = w_b \quad \text{in} \quad L^2.$$

On the other hand, setting

$$I_n := \left| \|u(s_n) - w_b\|_2^2 - \|u(t_n) - w_b\|_2^2 \right|, \qquad n \geq 1,$$

we have that

$$I_n = \left| \int_{t_n}^{s_n} \frac{d}{dt} \|u(t) - w_b\|_2^2 \, dt \right| \leq 2 \int_{t_n}^{s_n} \int_{\mathbb{R}^2} |u - w_b| |\partial_t u| \, dx dt$$
$$\leq \left(\int_{t_n}^{t_n+1} \|u(t) - w_b\|_2^2 \, dt \right)^{1/2} \left(\int_{t_n}^{t_n+1} \|\partial_t u(t)\|_2^2 \, dt \right)^{1/2}$$

and hence, we obtain that

$$\lim_{n \to \infty} I_n = 0.$$

Consequently,

$$\lim_{n \to \infty} u(t_n) = w_b \quad \text{in} \quad L^2$$

and, therefore, as this is valid along any sequence $\{t_n\}_{n\geq 1}$ approximating ∞ as $n \to \infty$, we find that

$$\lim_{t\to\infty} u(t) = w_b \quad \text{in} \quad L^2.$$

Moreover, thanks to Lemma 5.8, we have that

$$\sup_{t>0} \int_{\mathbb{R}^2} |x| u(t,x)\, dx < \infty$$

and, consequently, we also deduce that

$$\lim_{t\to\infty} u(t) = w_b \qquad \text{in} \;\; L^1.$$

Thus, it becomes apparent from the Nash inequality [42]

$$\|f\|_p \leq C_p \|f\|_1^{1/p} \|\nabla f\|_2^{1-1/p}, \qquad 1 \leq p < \infty,$$

that, for every $p \in [1, \infty)$,

$$\lim_{t \to \infty} u(t) = w_b \quad \text{in} \quad L^p. \tag{5.39}$$

In the case of $p = \infty$, we will use the interpolation inequality establishing that, for any $2 < q < \infty$, there exists a positive constant C_q, depending only on q, such that

$$\|f\|_\infty \leq C_q \|f\|_q^{1-2/q} \|\nabla f\|_q^{2/q}$$

for all $f \in W^{1,q}(\mathbb{R}^2)$. According to it, we find that

$$\|u(t) - w_b\|_\infty \leq C_q \|u(t) - w_b\|_q^{1-2/q} \|\nabla(u(t) - w_b)\|_q^{2/q} \tag{5.40}$$

for all $t \geq 3$ and $q \in (2, \infty)$. Therefore, (5.39) and (5.40) imply (5.39) for $p = \infty$:

$$\lim_{t \to \infty} u(t) = w_b \qquad \text{in} \quad L^\infty.$$

The proof is complete.