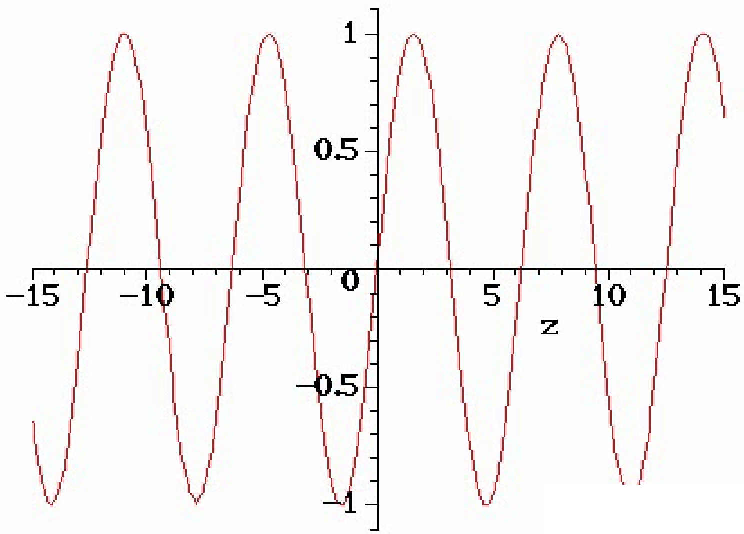
完全楕円積分と楕円関数

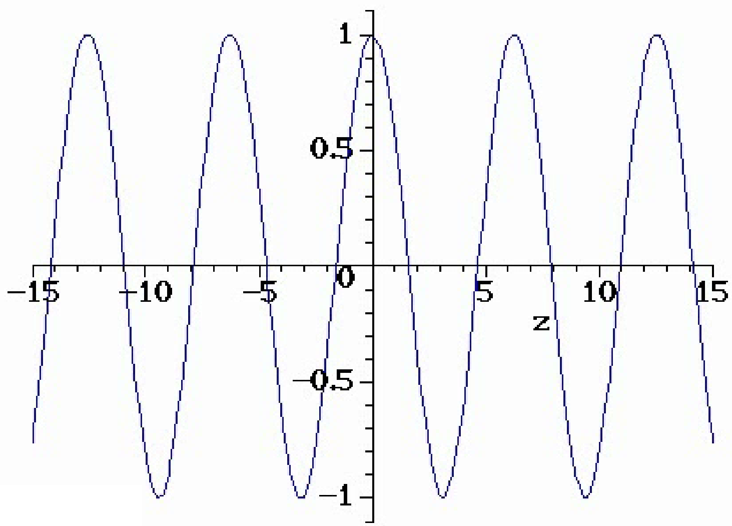
Jacobi's elliptic functions sn(z,k) and cn(z,k):

$$\begin{cases} \mathrm{sn}^{-1}(z,k) := \int_0^z \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, & z \in [0,1),\ k \in [0,1) \\ \mathrm{cn}(z,k) := \sqrt{1-\mathrm{sn}^2(z,k)}, & z \in (0, K(k)),\ k \in [0,1) \end{cases}$$

$\mathrm{sn}(z,k)$



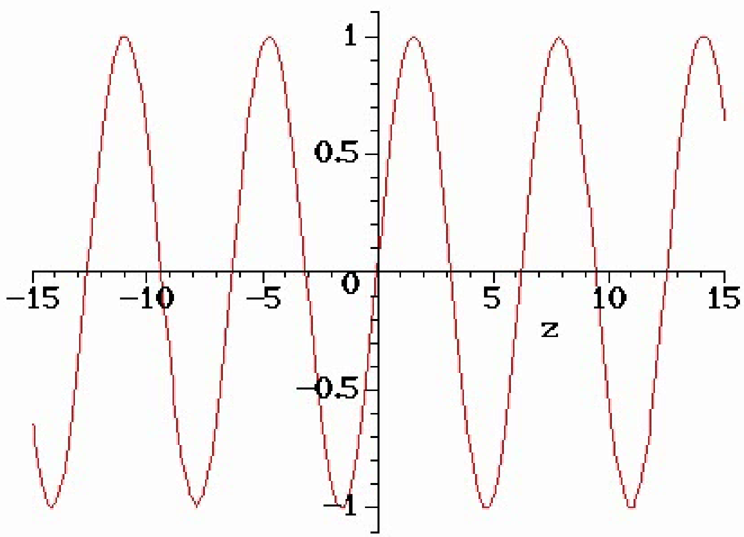
$\mathrm{cn}(z,k)$

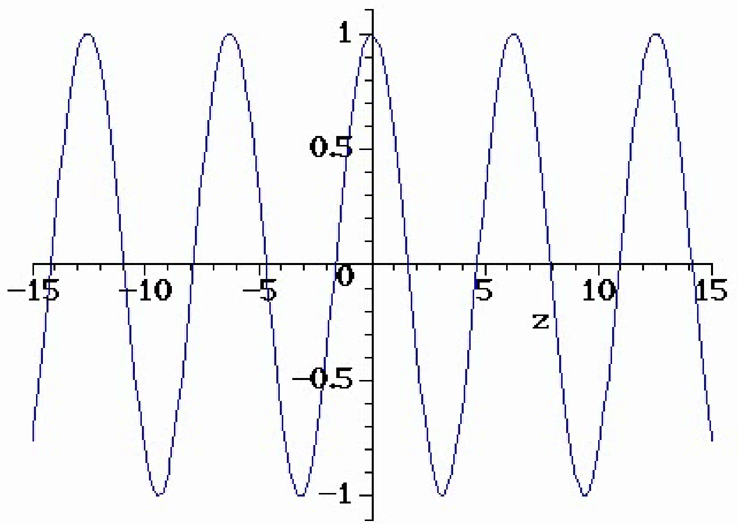


$$\begin{cases} \mathrm{sn}^{-1}(z,k) := \int_0^z \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, & z \in [0,1), k \in [0,1) \\ \mathrm{cn}(z,k) := \sqrt{1-\mathrm{sn}^2(z,k)}, & z \in (0, K(k)), k \in [0,1) \end{cases}$$

$\mathrm{sn}(z,k)$

$\mathrm{cn}(z,k)$





$$cn(z,k) \to \sec h(z,k) \quad as \quad k \to 1$$

Jacobi's elliptic functions sn(z,k) and cn(z,k):

$$\begin{cases} \mathrm{sn}^{-1}(z,k) := \int_0^z \dfrac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, \quad z \in [0,1), k \in [0,1) \\ \mathrm{cn}(z,k) := \sqrt{1-\mathrm{sn}^2(z,k)}, \quad z \in (0, K(k)), k \in [0,1) \end{cases}$$

1/4 period of $sn(z,k)$, $cn(z,k)$.

$$K(k) := \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

Complete elliptic integrals of the first kind

Notations: **complete elliptic integrals**

$$\begin{cases} K(k) := \int_0^{\frac{\pi}{2}} \dfrac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad k\in[0,1) \\ E(k) := \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\,d\varphi, \qquad k\in[0,1) \\ \Pi(\nu,k) := \int_0^{\frac{\pi}{2}} \dfrac{d\varphi}{(1+\nu\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}, \qquad k\in[0,1) \end{cases}$$

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

$$E(k) := \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

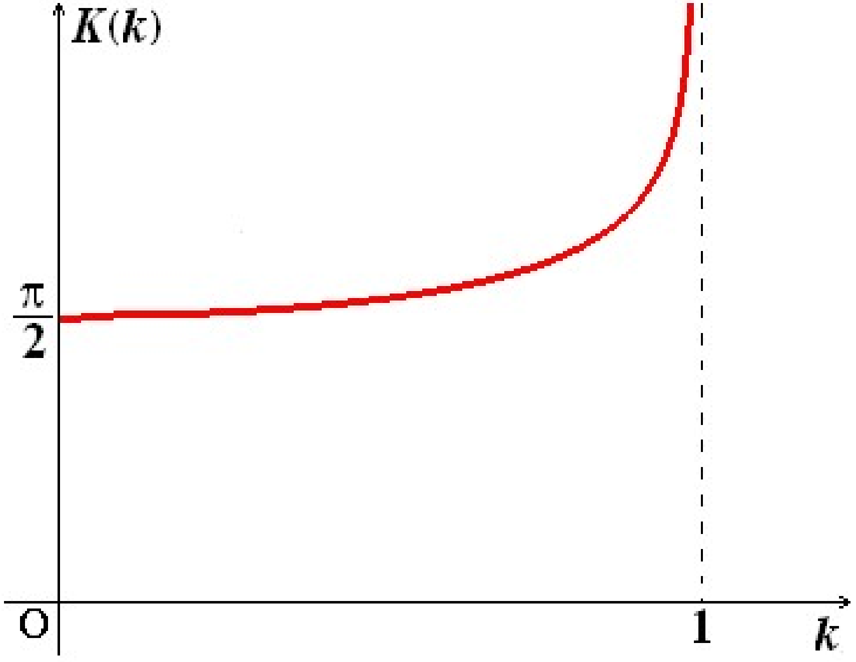
$$K(0) = \frac{\pi}{2}$$

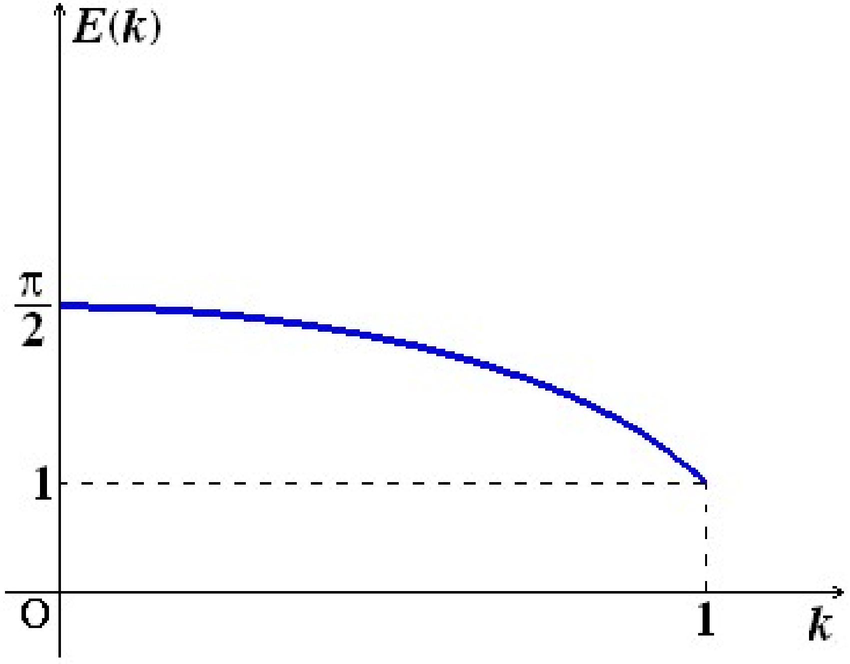
$$K(k) \to \infty,$$

$$(1-k^2)K(k) \to 0 \quad \text{as} \quad k \uparrow 1$$

$$E(0) = \frac{\pi}{2}$$

$$E(1) = 1$$

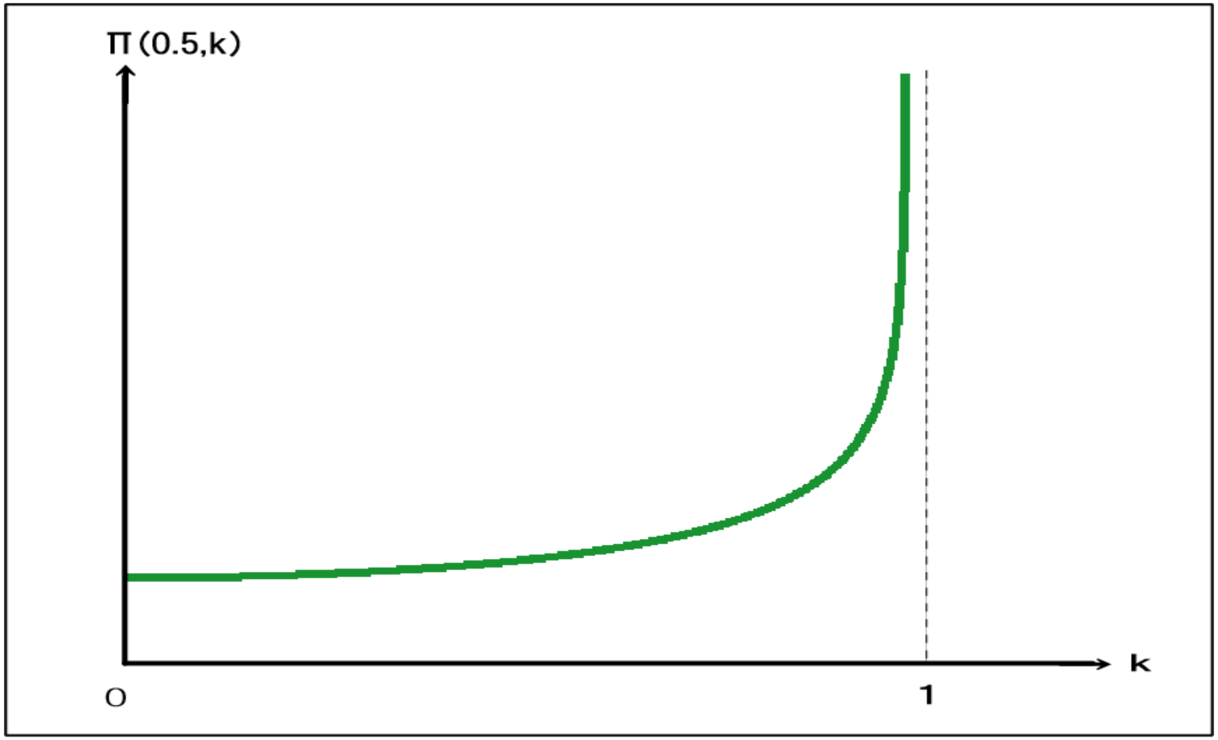




$\Pi(\nu,k)$ について

$$\Pi(\nu,k) := \int_0^{\frac{\pi}{2}} \frac{d\varphi}{(1+\nu\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}$$

$\nu = 0.5$ のとき



fundamental formula

$$\frac{d}{dk}K(k) = \frac{E(k)}{(1-k^2)k} - \frac{K(k)}{k},$$

$$\frac{d}{dk}E(k) = \frac{E(k)}{k} - \frac{K(k)}{k}$$

$$\frac{\partial}{\partial k}\Pi(n,k) = \frac{kE(k)}{(k^2+n)(1-k^2)} - \frac{k\Pi(n,k)}{(k^2+n)},$$

$$\frac{\partial}{\partial n}\Pi(n,k) = \frac{(k^2-n^2)\Pi(n,k)}{2(1+n)(k^2+n)n} - \frac{K(k)}{2(1+n)n} + \frac{E(k)}{2(1+n)(k^2+n)}$$

Proof of fundamental formula (only for easy ones)

$$K(k) := \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\varphi}}\, d\varphi$$

$$E(k) := \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\, d\varphi$$

$$\frac{d}{dk}K(k) = \frac{E(k)}{(1-k^2)k} - \frac{K(k)}{k},$$

$$\frac{d}{dk}E(k) = \frac{E(k)}{k} - \frac{K(k)}{k}$$

$$\begin{aligned}
\frac{dE(K)}{dk} &= \frac{d}{dk}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta \\
&= \int_0^{\pi/2}\frac{\partial}{\partial k}\sqrt{1-k^2\sin^2\theta}d\theta = \int_0^{\pi/2}\frac{-k\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}} \\
&= \frac{1}{k}\left(\int_0^{\pi/2}\frac{1-k^2\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta - \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\right) \\
&= \frac{1}{k}(E(k)-K(k))
\end{aligned}$$

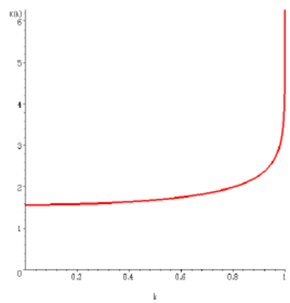
$$\begin{aligned}
\frac{dK(K)}{dk} = \frac{d}{dk}\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta &= \int_0^{\pi/2}\frac{\partial}{\partial k}\left(\frac{1}{\sqrt{1-k^2\sin^2\theta}}\right)d\theta \\
&= \int_0^{\pi/2}\frac{k\sin^2\theta}{(1-k^2\sin^2\theta)^{3/2}}d\theta = \int_0^{\pi/2}\frac{k}{1-k^2}\frac{d}{d\theta}\left(-\frac{\cos\theta}{\sqrt{1-k^2\sin^2\theta}}\right)\sin\theta d\theta \\
&= \frac{k}{1-k^2}\left[-\frac{\cos\theta\sin\theta}{\sqrt{1-k^2\sin^2\theta}}\right]_0^{\pi/2} + \frac{k}{1-k^2}\int_0^{\pi/2}\frac{\cos^2\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta \\
&= \frac{k}{1-k^2}\int_0^{\pi/2}\frac{1}{k^2}\frac{1-k^2\sin^2\theta-(1-k^2)}{\sqrt{1-k^2\sin^2\theta}}d\theta \\
&= \frac{1}{k(1-k^2)}\left(\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}d\theta-(1-k^2)\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta\right) \\
&= \frac{1}{k(1-k^2)}(E(k)-(1-k^2)K(k)).
\end{aligned}$$

Complete elliptic integrals

the first kind complete elliptic integral

$$K(k) := \int_0^1 \frac{1}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}\, d\xi = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\varphi}}$$

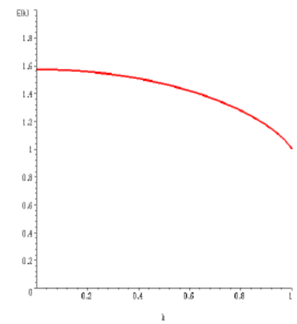
$K(0) = \pi/2,$ $\qquad$ $K(k) \to \infty$ as $k \to 1$



the second kind complete elliptic integral

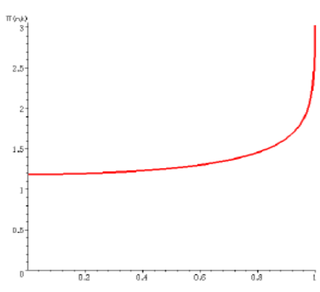
$$E(k) := \int_0^1 \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}\, d\xi = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\, d\varphi,$$

$E(0) = \pi/2,$ $\qquad$ $E(1) = 1$



the third kind complete elliptic integral

$$\Pi(n,k) = \int_0^1 \frac{1}{(1+n\xi^2)\sqrt{(1-\xi^2)(1-k^2\xi^2)}} d\xi$$



Appendix A. Fundamental results of elliptic integrals

Lemma A.1. *Let $k \in (0, 1)$ and $\nu \neq 0, -1, -k^2$. Then*

(i) $\frac{dE}{dk}(k) = \frac{E(k)-K(k)}{k}$,

(ii) $\frac{dK}{dk}(k) = \frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}$,

(iii) $\frac{\partial \Pi}{\partial k}(\nu, k) = \frac{k(E(k)-(1-k^2)\Pi(\nu,k))}{(k^2+\nu)(1-k^2)}$,

(iv) $\frac{\partial \Pi}{\partial \nu}(\nu, k) = -\frac{K(k)}{2\nu(1+\nu)} + \frac{E(k)}{2(1+\nu)(k^2+\nu)} + \frac{(k^2-\nu^2)\Pi(\nu,k)}{2\nu(1+\nu)(k^2+\nu)}$.

Lemma A.2. *For every $k \in (0, 1)$,*

$$(1-k^2)K(k) < E(k) < \left(1 - \frac{1}{2}k^2\right)K(k).$$

Lemma A.3. *Let $k \in (0, 1)$ and K be the elliptic integral of the first kind. Then,*

$$\lim_{k\to 1}\left(K(k) - \log\frac{1}{\sqrt{1-k^2}} - 2\log 2\right) = 0.$$

In addition to Lemmas A.1–A.3, we will use several asymptotic formulas for the elliptic integral $\Pi(\nu, k)$. The changes of variables $s = \tau/\sqrt{1+\nu+\tau^2}$ and $s = \tau/\sqrt{1+\tau^2}$ leads us to

$$\sqrt{1+\nu}\,\Pi(\nu, k) = \int_0^{+\infty} \frac{1}{1+\tau^2}\sqrt{\frac{1+\nu+\tau^2}{1+\nu+(1-k^2)\tau^2}}\,d\tau \tag{A.1}$$

and

$$\Pi(\nu, k) = \int_0^{+\infty} \frac{1+\tau^2}{(1+(1+\nu)\tau^2)\sqrt{1+\tau^2}\sqrt{1+(1-k^2)\tau^2}}d\tau, \tag{A.2}$$

respectively.

Remark A.1. In the case of $\nu = 0$, both (A.1) and (A.2) imply that

$$K(k) = \int_0^{+\infty} \frac{1}{\sqrt{1+\tau^2}\sqrt{1+(1-k^2)\tau^2}}d\tau,$$

which is useful to show the asymptotic formula of K as in Lemma A.3.

Lemma A.4. *Let* $k \in (0, 1)$ *and* $\nu > -1$. *Then, the following* (i) *and* (ii) *hold*:

(i) $\lim_{\nu \to -1} \sqrt{1+\nu}\, \Pi(\nu, k) = \frac{\pi}{2\sqrt{1-k^2}}$,
(ii) $\lim_{\nu \to +\infty} \sqrt{1+\nu}\, \Pi(\nu, k) = \frac{\pi}{2}$

Moreover, we prepare more special asymptotic formulas corresponding to (i) and (ii) of Lemma A.4.

Lemma A.5. *Let* $k \in (0, 1)$. *Suppose that* ν *is a continuous function on* $(0, 1)$ *with* $-1 < \nu(k) < -k^2$ *for* $k \in (0, 1)$. *Assume there exists* $\nu^* \in [0, 1]$ *such that*

$$\lim_{k \to 1} \frac{1+\nu(k)}{1-k^2} = \nu^*.$$

Then, for each $\nu^* \in [0, 1]$,

$$\lim_{k \to 1} \sqrt{-\big(1+\nu(k)\big)\big(k^2+\nu(k)\big)} \cdot \Pi\big(\nu(k), k\big) = \frac{\pi}{2} - \tan^{-1}\sqrt{\frac{\nu^*}{1-\nu^*}}.$$

Example. Limit of complete integral of the third kind

Assume $0 \le k < 1, -1 < u < 1$, $\lim_{k\uparrow 1} u = 0$

$$\lim_{k\uparrow 1} \sqrt{1-u^2}(1-k^2)\Pi(\frac{1}{2}(1-u)(1-k^2)-1, k) = \frac{\pi}{2}$$

Lemma A.6. *Set* $J(\nu, k) := \sqrt{1+\nu}\,\Pi(\nu, k) - \frac{1}{\sqrt{1+\nu}} K(k)$. *Then,*

$$\lim_{\nu \to +\infty, k \to 1} J(\nu, k) = \frac{\pi}{2}.$$

Let us introduce an elliptic integral of the form:

$$\tilde{\Pi}(a, b, k) := \int_0^1 \frac{1}{\sqrt{(1-s^2)(1-k^2 s^2)}[a + (b - s^2)^2]} ds, \tag{A.3}$$

where $a > 0$ and $b \in (0, 1)$. It is easy to see that $\tilde{\Pi}$ can be expressed by $\Pi((b + \sqrt{-a})/(a + b^2), k)$ and $\Pi((b - \sqrt{-a})/(a + b^2), k)$. We derive some asymptotic formulas for $\tilde{\Pi}$ as follows.

Lemma A.7. *Suppose* $a > 0$ *and* $b, b_0 \in (0, 1)$. *Then for each* $k \in (0, 1)$,

$$\lim_{a \to 0, b \to b_0} \sqrt{a}\,\tilde{\Pi}(a, b, k) = \frac{\pi}{2\sqrt{b_0(1-b_0)(1-k^2 b_0)}}.$$

Lemma A.8. *Suppose $a > 0$, $b, b_0 \in (0, 1)$ and $k \in (0, 1)$. Set*

$$\tilde{J}(a, b, k) := \sqrt{a}\tilde{\Pi}(a, b, k) - \frac{\sqrt{a}}{a + (1-b)^2} K(k).$$

Then,

$$\lim_{a \to 0, b \to b_0, k \to 1} \tilde{J}(a, b, k) = \frac{\pi}{2\sqrt{b_0}(1 - b_0)}.$$

Tohru Wakasa, Exact eigenvalues and eigenfunctions associated with linearization for Chafee–Infante problem, Funkcial. Ekvac. 49 (2) (2006) 321–336.

Tohru Wakasa, Shoji Yotsutani, Representation formulas for some 1-dimensional linearized eigenvalue problems, Commun. Pure Appl. Anal. 7 (4) (2008) 745–763.

Tohru Wakasa, Shoji Yotsutani, Asymptotic profiles of eigenfunctions for some 1-dimensional linearized eigenvalue problems, Commun. Pure Appl. Anal. 9 (2) (2010) 539–561.

Tohru Wakasa, Shoji Yotsutani, Limiting classifications on linearized eigenvalue problem for 1-dimensional Allen–Cahn equation II—asymptotic profiles of eigenfunctions, in preparation.

$\Pi(hs, \sqrt{h})$ のとりあつかいについて

$\boxed{\Pi(hs, \sqrt{h})}$ は微分すると $\Pi(hs, \sqrt{h}), K(\sqrt{h}), E(\sqrt{h})$ がでる.

$$\frac{\partial}{\partial s}\Pi(hs,\sqrt{h}) = \frac{1}{2}\frac{(1-hs^2)\Pi(hs,\sqrt{h}) - sE(\sqrt{h}) - (1-s)K(\sqrt{h})}{s(1-s)(1-sh)}$$

$$\frac{\partial}{\partial h}\Pi(hs,\sqrt{h}) = \frac{1}{2}\frac{sh(1-h)\Pi(hs,\sqrt{h}) + E(\sqrt{h}) - (1-h)K(\sqrt{h})}{h(1-h)(1-sh)}$$

$\Pi(hs, \sqrt{h})$ のとりあつかいについて

$\Pi(hs, \sqrt{h})$ は微分すると $\Pi(hs,\sqrt{h}), K(\sqrt{h}), E(\sqrt{h})$ がでる.

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$$\frac{\partial}{\partial h}\Pi(hs,\sqrt{h}) = \frac{1}{2}\frac{sh(1-h)\Pi(hs,\sqrt{h}) + E(\sqrt{h}) - (1-h)K(\sqrt{h})}{h(1-h)(1-sh)}$$

$\Pi(hs, \sqrt{h})$ を消す方法あり！

$\Pi(hs, \sqrt{h})$ のとりあつかいについて

$\boxed{\Pi(hs, \sqrt{h})}$ は微分すると $\Pi(hs,\sqrt{h}), K(\sqrt{h}), E(\sqrt{h})$ がでる.

$$\frac{\partial}{\partial s}\Pi(hs,\sqrt{h}) = \frac{1}{2}\frac{(1-hs^2)\Pi(hs,\sqrt{h}) - sE(\sqrt{h}) - (1-s)K(\sqrt{h})}{s(1-s)(1-sh)}$$

$$\frac{\partial}{\partial h}\Pi(hs,\sqrt{h}) = \frac{1}{2}\frac{sh(1-h)\Pi(hs,\sqrt{h}) + E(\sqrt{h}) - (1-h)K(\sqrt{h})}{h(1-h)(1-sh)}$$

$\Pi(hs, \sqrt{h})$ を消す方法あり！

$$\boxed{\frac{2\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\cdot\Pi(hs,\sqrt{h})}$$

$$\frac{\partial}{\partial s}\left(\frac{2\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\cdot\Pi(hs,\sqrt{h})\right) = -\frac{sE(\sqrt{h}) + (1-s)K(\sqrt{h})}{s\sqrt{s(1-s)(1-sh)}}$$

$$\frac{\partial}{\partial h}\left(\frac{2\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\cdot\Pi(hs,\sqrt{h})\right) = \frac{(1-s)\Big(E(\sqrt{h}) - (1-h)K(\sqrt{h})\Big)}{h(1-h)\sqrt{s(1-s)(1-sh)}}$$

$\Pi(hs, \sqrt{h})$ のとりあつかいについて

$\Pi(hs, \sqrt{h})$ は微分すると $\Pi(hs,\sqrt{h}), K(\sqrt{h}), E(\sqrt{h})$ がでる.

$$\frac{\partial}{\partial s}\Pi(-hs,\sqrt{h}) = \frac{1}{2}\frac{(1-hs^2)\Pi(-hs,\sqrt{h}) - sE(\sqrt{h}) - (1-s)K(\sqrt{h})}{s(1-s)(1-sh)}$$

$$\frac{\partial}{\partial h}\Pi(-hs,\sqrt{h}) = \frac{1}{2}\frac{sh(1-h)\Pi(-hs,\sqrt{h}) + E(\sqrt{h}) - (1-h)K(\sqrt{h})}{h(1-h)(1-sh)}$$

$\Pi(hs, \sqrt{h})$ を消す方法あり！

$$\frac{2\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\cdot\Pi(-hs,\sqrt{h})$$

K.Kosugi, Y. Morita and Y. Yotsutani, DCDS 19(2007),

$$\frac{\partial}{\partial s}\left(\frac{2\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\cdot\Pi(-hs,\sqrt{h})\right) = -\frac{sE(\sqrt{h}) + (1-s)K(\sqrt{h})}{s\sqrt{s(1-s)(1-sh)}}$$

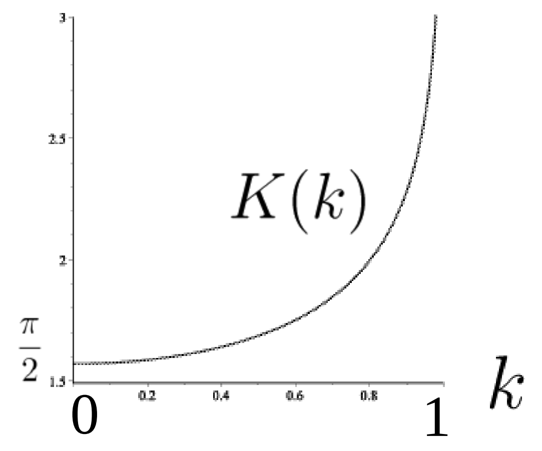
$$\frac{\partial}{\partial h}\left(\frac{2\sqrt{(1-s)(1-sh)}}{\sqrt{s}}\cdot\Pi(-hs,\sqrt{h})\right) = \frac{(1-s)\left(E(\sqrt{h}) - (1-h)K(\sqrt{h})\right)}{h(1-h)\sqrt{s(1-s)(1-sh)}}$$

完全楕円積分 E/K の近似式について

第１種完全楕円積分

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\phi}} d\phi$$

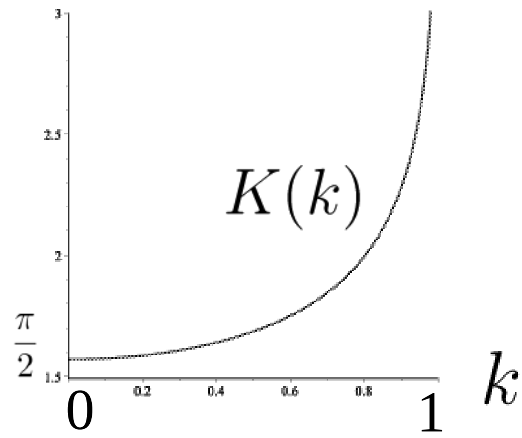
$K(0)=\dfrac{\pi}{2}$,　$k\to 1$ のとき $K(k)\to\infty$



第１種完全楕円積分

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\phi}}d\phi$$

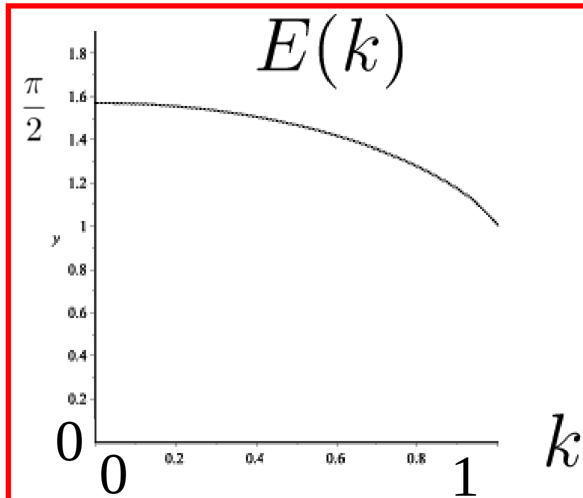
$K(0)=\frac{\pi}{2}$,　$k \to 1$ のとき $K(k) \to \infty$



第２種完全楕円積分

$$E(k) := \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\phi}\,d\phi$$

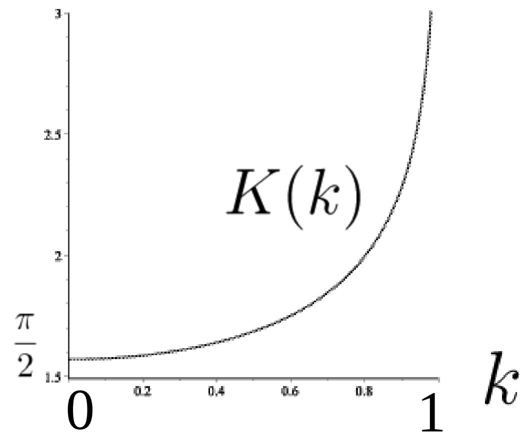
$E(0)=\frac{\pi}{2}$, $E(1)=1$



第１種完全楕円積分

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\phi}}d\phi$$

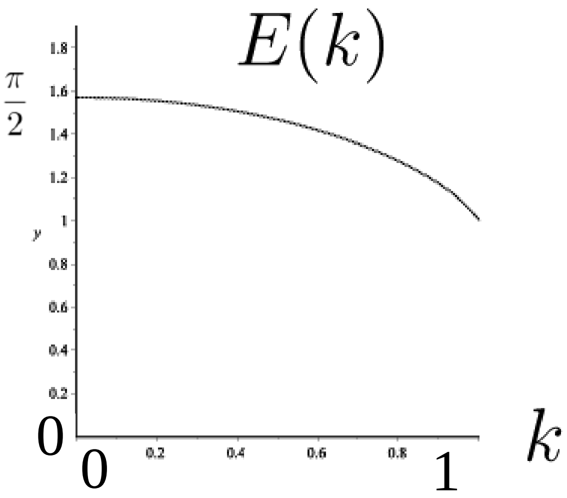
$K(0) = \frac{\pi}{2}$, $k \to 1$ のとき $K(k) \to \infty$



第２種完全楕円積分

$$E(k) := \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\phi}d\phi$$

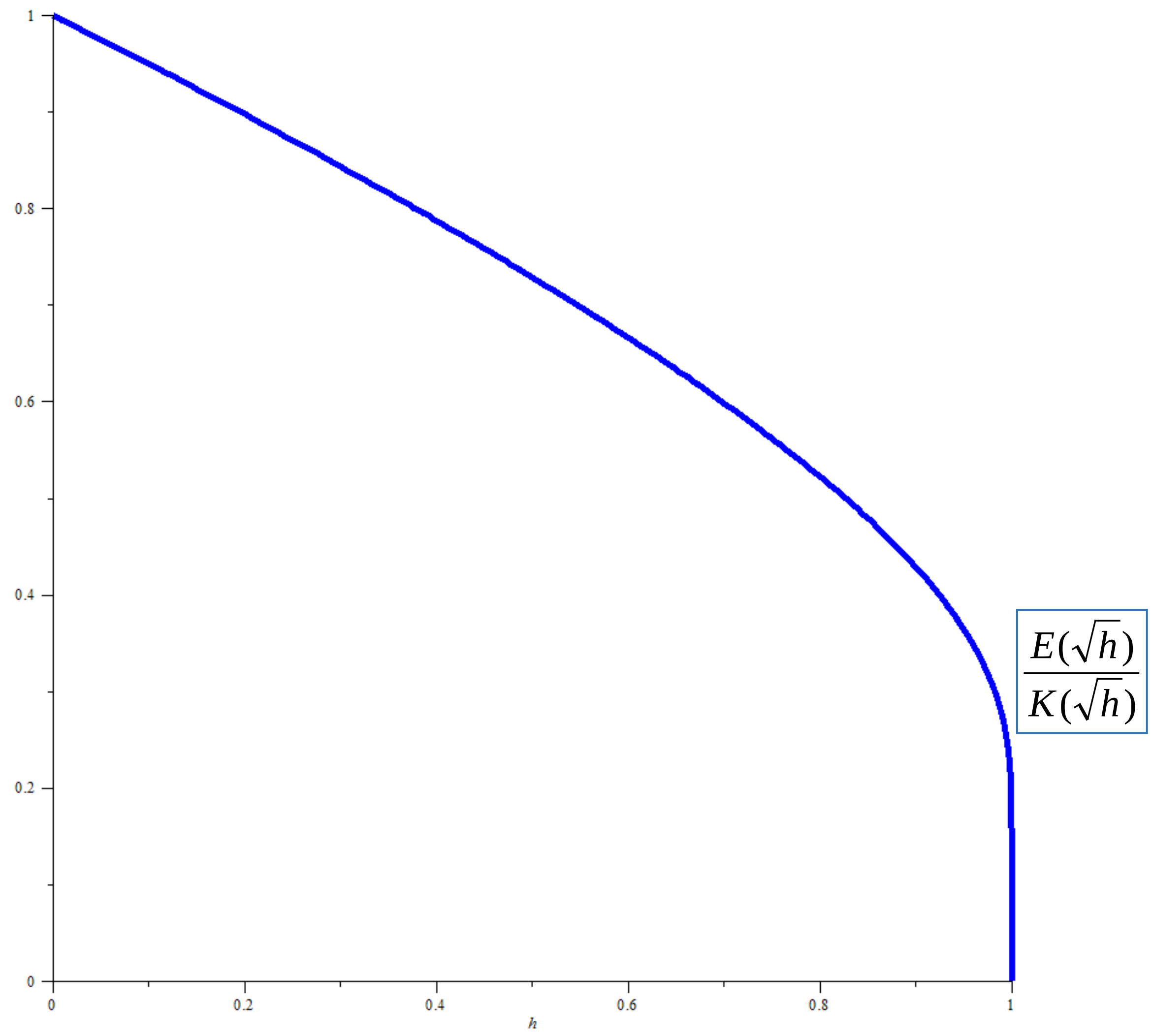
$E(0) = \frac{\pi}{2}$, $E(1) = 1$



$$\frac{d}{dk}E(k) = \frac{E(k)}{k} - \frac{K(k)}{k}$$

$$\frac{d}{dk}K(k) = \frac{E(k)}{(1-k^2)k} - \frac{K(k)}{k}$$

$\frac{E(\sqrt{h})}{K(\sqrt{h})}$ の上下からの評価式



$$\frac{E(\sqrt{h})}{K(\sqrt{h})}$$ の上下からの評価式

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} < 1$$

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} < 1 - \frac{h}{2}$$

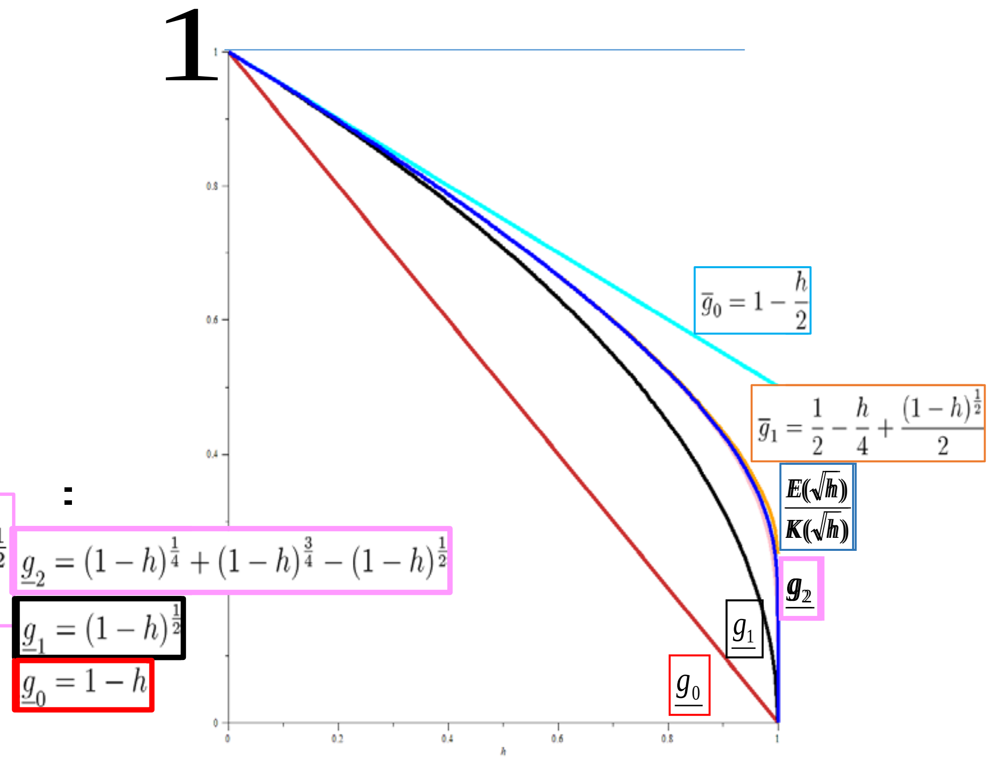
$$\frac{E(\sqrt{h})}{K(\sqrt{h})} < (\frac{1}{2} - \frac{1}{4}h + \frac{1}{2}\sqrt{1-h})$$

⋮

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} > (1-h)^{\frac{1}{4}} + (1-h)^{\frac{3}{4}} - (1-h)^{\frac{1}{2}}$$

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} > \sqrt{1-h}$$

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} > 1 - h$$



$\frac{E(\sqrt{h})}{K(\sqrt{h})}$ の上下からの評価式　村井－松本－四ッ谷（2010） ← ガウス（1816）

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} < 1$$

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} < 1 - \frac{h}{2}$$

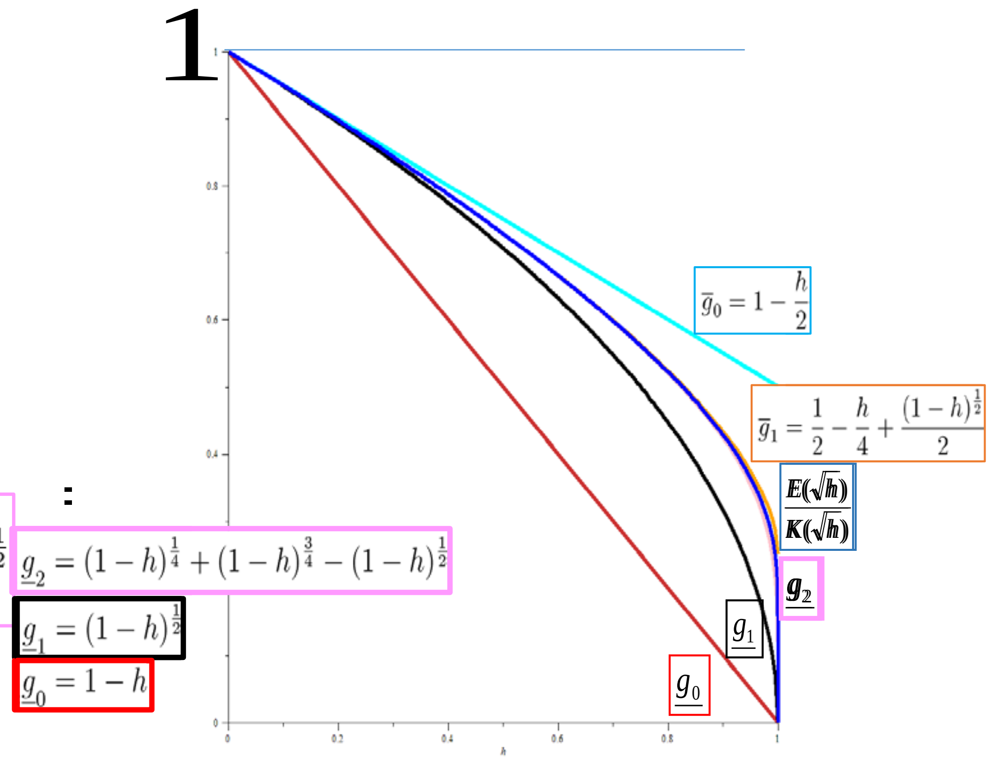
$$\frac{E(\sqrt{h})}{K(\sqrt{h})} < (\frac{1}{2} - \frac{1}{4}h + \frac{1}{2}\sqrt{1-h})$$

⋮

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} > (1-h)^{\frac{1}{4}} + (1-h)^{\frac{3}{4}} - (1-h)^{\frac{1}{2}}$$

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} > \sqrt{1-h}$$

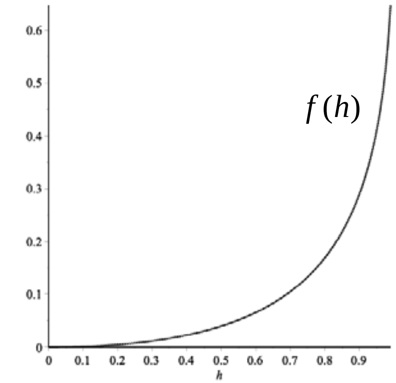
$$\frac{E(\sqrt{h})}{K(\sqrt{h})} > 1-h$$



命題　**[この場合は直接証明が簡単にできる]**

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} > \sqrt{1-h} \quad (0 < h < 1).$$

証明



$f(h) := E(\sqrt{h}) - (\sqrt{1-h})K(\sqrt{h})$ とおく.

$$f'(h) = \frac{1}{2}\cdot\frac{\frac{E(\sqrt{h})}{\sqrt{h}} - \frac{K(\sqrt{h})}{\sqrt{h}}}{\sqrt{h}} + \frac{1}{2}\cdot\frac{K(\sqrt{h})}{\sqrt{1-h}} - \frac{1}{2}\cdot\frac{\sqrt{(1-h)}\left(\frac{E(\sqrt{h})}{(1-h)\ \sqrt{h}} - \frac{K(\sqrt{h})}{\sqrt{h}}\right)}{\sqrt{h}}$$

$$= \frac{1}{2}\cdot\frac{(\sqrt{1-h}-1)(E(\sqrt{h}) - K(\sqrt{h}))}{\sqrt{1-h}\cdot h}$$

$$= \frac{1}{2}\cdot\frac{(\sqrt{1-h}-1)(\sqrt{1-h}+1)(E(\sqrt{h}) - K(\sqrt{h}))}{\sqrt{1-h}\cdot h(\sqrt{1-h}+1)}$$

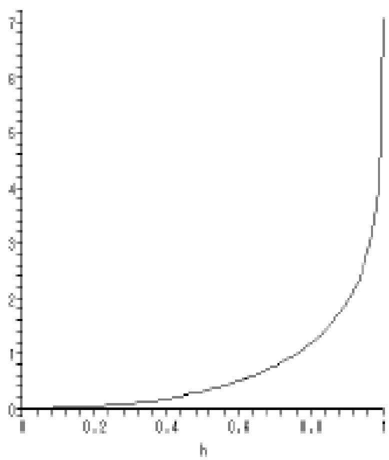
$$= \frac{1}{2}\frac{K(\sqrt{h}) - E(\sqrt{h})}{(1-h+\sqrt{1-h})} \ > \ 0 \ \ (0 < h < 1)$$

$$f(0) = E(0) - K(0) = \frac{\pi}{2} - \frac{\pi}{2} = 0$$

6 Inequalities of complete elliptic integrals

Problem 6.1. *Prove the following inequality*

$$-3E(\sqrt{h})^2 + 2(2-h)K(\sqrt{h})E(\sqrt{h}) - (1-h)K(\sqrt{h})^2 > 0 \ (0 < h < 1).$$



Usual answer

$$f(h) := -3E(\sqrt{h})^2 + 2(2-h)K(\sqrt{h})E(\sqrt{h}) - (1-h)K(\sqrt{h})^2.$$

By $K(0) = E(0) = \pi/2$,

$$\begin{aligned} f(0) &= -3E(0)^2 + 4K(0)E(0) - K(0)^2 \\ &= -3\left(\frac{\pi}{2}\right)^2 + 4\cdot\frac{\pi}{2}\cdot\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^2 = 0. \end{aligned}$$

By

$$\begin{aligned} \frac{d}{dk}K(k) &= \frac{E(k)-(1-k^2)K(k)}{k(1-k^2)}, \\ \frac{d}{dk}E(k) &= \frac{E(k)-K(k)}{k}, \end{aligned} \tag{5}$$

we obtain

$$\begin{aligned} \frac{d}{dh}K(\sqrt{h}) &= \frac{1}{2}\cdot\frac{E(\sqrt{h})-(1-h)K(\sqrt{h})}{h(1-h)}, \\ \frac{d}{dk}E(\sqrt{h}) &= \frac{1}{2}\cdot\frac{E(\sqrt{h})-K(\sqrt{h})}{h}. \end{aligned} \tag{6}$$

Usual answer

$$f(h) := -3E(\sqrt{h})^2 + 2(2-h)K(\sqrt{h})E(\sqrt{h}) - (1-h)K(\sqrt{h})^2.$$

By $K(0) = E(0) = \pi/2$,

$$\begin{aligned} f(0) &= -3E(0)^2 + 4K(0)E(0) - K(0)^2 \\ &= -3\left(\frac{\pi}{2}\right)^2 + 4\cdot\frac{\pi}{2}\cdot\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^2 = 0. \end{aligned}$$

By

$$\begin{aligned} \frac{d}{dk}K(k) &= \frac{E(k) - (1-k^2)K(k)}{k(1-k^2)}, \\ \frac{d}{dk}E(k) &= \frac{E(k) - K(k)}{k}, \end{aligned} \tag{5}$$

we obtain

$$\begin{aligned} \frac{d}{dh}K(\sqrt{h}) &= \frac{1}{2}\cdot\frac{E(\sqrt{h}) - (1-h)K(\sqrt{h})}{h(1-h)}, \\ \frac{d}{dk}E(\sqrt{h}) &= \frac{1}{2}\cdot\frac{E(\sqrt{h}) - K(\sqrt{h})}{h}. \end{aligned} \tag{6}$$

Hence

$$\begin{aligned} \frac{d}{dh}f(h) =& -3E(\sqrt{h})\left(\frac{E(\sqrt{h}) - K(\sqrt{h})}{h}\right) - 2K(\sqrt{h})E(\sqrt{} \\ & +2(2-h)\left(\frac{E(\sqrt{h}) - (1-h)K(\sqrt{h})}{2h(1-h)}\right)E(\sqrt{h}) \\ & +2(2-h)K(\sqrt{h})\left(\frac{E(\sqrt{h}) - K(\sqrt{h})}{2h}\right) \\ & +K(\sqrt{h})^2 \\ & -(1-h)K(\sqrt{h})\left(\frac{E(\sqrt{h}) - (1-h)K(\sqrt{h})}{h(1-h)}\right) \\ =& \frac{(E(\sqrt{h}) - (1-h)K(\sqrt{h}))}{h(1-h)}\cdot((1-h)K(\sqrt{h}) - (1-2h)I \end{aligned}$$

We note that

$$\begin{aligned} & E(\sqrt{h}) - (1-h)K(\sqrt{h}) \\ &= \int_0^{\pi/2}\sqrt{1-h\sin^2\varphi}\,d\varphi - \int_0^{\pi/2}\frac{1-h}{\sqrt{1-h\sin\varphi^2}}\,d\varphi \\ &= h\int_0^{\pi/2}\frac{\cos^2\varphi}{\sqrt{1-h\sin^2\varphi}}\,d\varphi > 0. \end{aligned}$$

and

$$\begin{aligned} & (1-h)K(\sqrt{h}) - (1-2h)E(\sqrt{h}) \\ &= (1-h)\big(K(\sqrt{h}) - E(\sqrt{h})\big) + hE(\sqrt{h}) > 0. \end{aligned}$$

Therefore

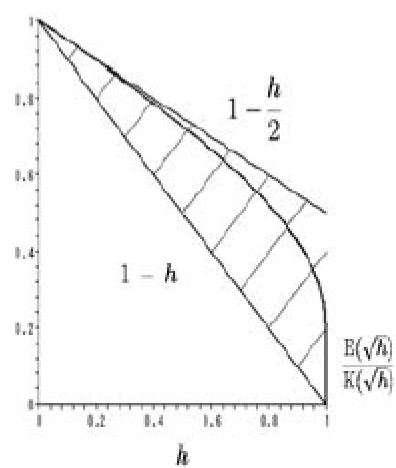
$$\frac{d}{dh}f(h) > 0 \quad (0 < h < 1).$$

Answer (New method)

$$-3\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)^2 + 2(2-h)\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right) - (1-h) > 0 \quad (0 < h < 1)$$

We note that the following inequality holds (easy to prove):

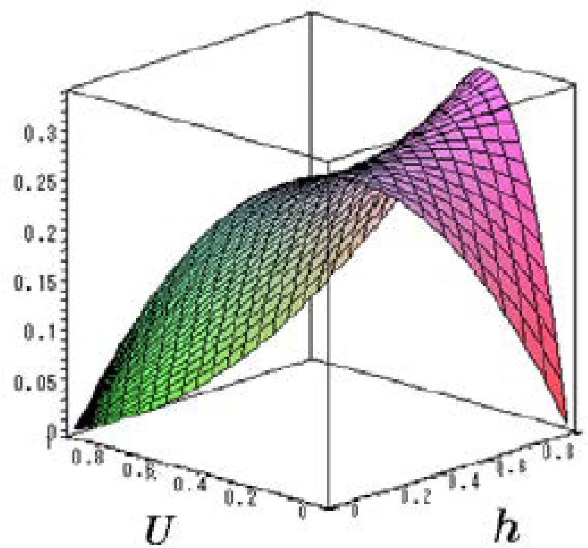
$$1 - h < \frac{E(\sqrt{h})}{K(\sqrt{h})} < 1 - \frac{h}{2} \quad (0 < h < 1)$$

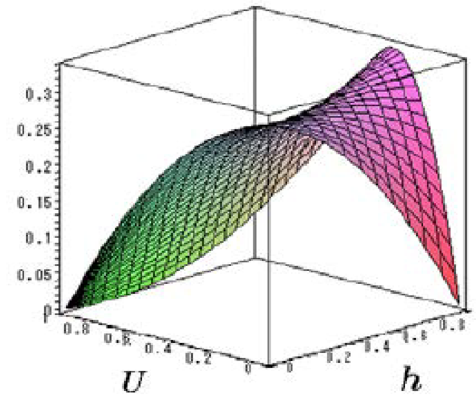


We may show that $f(h, U) > 0$ on D, where

$$D := \left\{ (h, U)\,;\, 0 < h < 1,\, 1 - h < U < 1 - \frac{h}{2} \right\},$$

$$f(h, U) := -3U^2 + 2(2 - h)U - (1 - h).$$





$$f\left(h, 1-\frac{h}{2}\right) = \frac{h^2}{4} > 0, \qquad (0 < h < 1),$$

$$f(h, 1-h) = h(1-h) > 0 \quad (0 < h < 1),$$

$$f(1, U) = U(2-3U) > 0$$

Moreover,

$$\frac{\partial f}{\partial h} = -2U + 1 = 0, \quad \frac{\partial f}{\partial U} = -6U + 2(2-h) = 0$$

implies

$$(h, u) = (1/2, 1/2) \notin D.$$

Consequently,

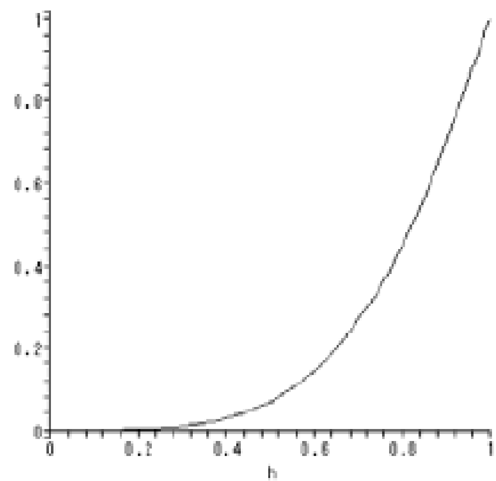
$$f(h, U) > 0 \quad in\ D.$$

Problem 6.2. *Prove the following inequality*

$$(h^2 - h + 1)E(\sqrt{h})^4 - 2(1-h)(2-h)E(\sqrt{h})^3K(\sqrt{h})$$

$$+6(1-h)^2E(\sqrt{h})^2K(\sqrt{h})^2 - 2(2-h)(1-h)^2E(\sqrt{h})K(\sqrt{h})^3$$

$$+(1-h)^3K(\sqrt{h})^4 > 0. \qquad (0 < h < 1)$$

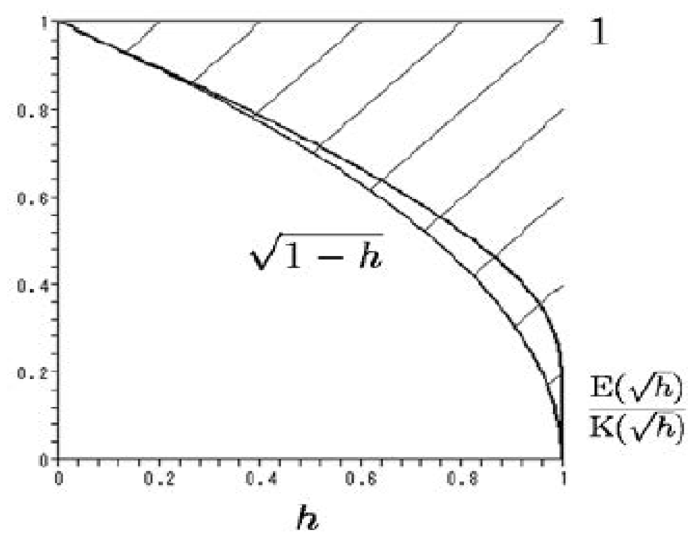


Answer We may show

$$(h^2-h+1)\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)^4-2(1-h)(2-h)\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)^3$$
$$+6(1-h)^2\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)^2-2(2-h)(1-h)^2\left(\frac{E(\sqrt{h})}{K(\sqrt{h})}\right)$$
$$+(1-h)^3>0 \quad (0<h<1).$$

It hold that

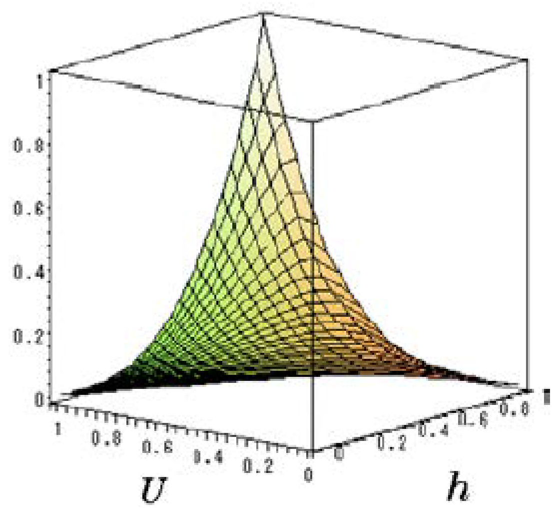
$$\sqrt{1-h}<\frac{E(\sqrt{h})}{K(\sqrt{h})}<1 \quad (0<h<1).$$



We may show in $f(h,U) > 0$ in D, where

$$D := \left\{(h,U);\ 0 < h < 1,\ \sqrt{1-h} < U < 1\right\},$$

$$f(h,U) := (h^2 - h + 1)U^4 - 2(1-h)(2-h)U^3 + 6(1-h)^2U^2 \\ -2(2-h)(1-h)^2U + (1-h)^3.$$



We have

$$f(h,1) = h^3 > 0 \qquad (0 < h < 1),$$

$$f(h,\sqrt{1-h}) = (1-h)^2(1-\sqrt{1-h})^4 > 0 \qquad (0 < h < 1),$$

$$f(1,U) = U^4 > 0 \quad (0 < U < 1).$$

Let us investigate the following system of algebraic equation

$$\begin{aligned} f_h &= (2h-1)U^4 + 2(3-2h)U^3 - 12(1-h)U^2 \\ &\quad +2(1-h)(5-3h)U - 3(1-h)^2 = 0, \end{aligned}$$

$$\begin{aligned} f_U &= 4(h^2-h+1)U^3 - 6(1-h)(2-h)U^2 \\ &\quad +12(1-h)^2U - 2(2-h)(1-h)^2 = 0. \end{aligned}$$

We will show that there exists no solution in

$$\{(h,U) : 0 < h < 1, 0 < U < 1\}.$$

By using the Buchburger algorism (Groebner basis), we see that the above equation is equivalent to the following system with three equations.:

$$(1-2h)h^3(1-h)^3 \cdot (125h^4 - 250h^3 + 621h^2 - 496h + 128) = 0,$$

$$h(1-h)(4250h^7 - 14875h^6 + 34989h^5 - 50285h^4 \\ +34237h^3 - 8508h^2 - 192h + 384 - 384U) = 0,$$

$$864U^3 - 2592(1-h)U^2 + 2592(1-h)U \\ +(1-h)^2 \cdot (22000h^7 - 55000h^6 + 125046h^5 \\ -132569h^4 + 42251h^3 - 864h^2 - 1728h - 864) = 0.$$

We see from the Sturm theorem for zeros that

$$125h^4 - 250h^3 + 621h^2 - 496h + 128 > 0 \ (0 < h < 1).$$

Thus $h = 1/2$, which implies $U = 1/2$. Cnsequently there exists no solution in D since $(h, U) = (1/2, 1/2) \notin D$.

7 Comparison function for E/K and $1/K$

Theorem 7.1 ([3]). *Set* $\overline{g}_n(h)$, $\underline{g}_n(h)$ *by*

$$\begin{aligned}\overline{g}_n(h) &:= 1-\sum_{m=0}^{n} 2^{m-1}c_m(h)^2,\\ \underline{g}_n(h) &:= 1-\sum_{m=0}^{n} 2^{m-1}c_m(h)^2-2^{n-1}c_n(h)^2.\end{aligned}$$

Then

$$\underline{g}_n(h) \leq \frac{E(\sqrt{h})}{K(\sqrt{h})} \leq \overline{g}_n(h) \quad on\ [0,1] \qquad (n=0,1,2,\cdots).,$$

and

$$\begin{aligned}&left \leq becomes = \iff h=0,1,\\ &right \leq becomes = \iff h=0,\end{aligned}$$

Moreover,

$$\underline{g}_n(h),\ \overline{g}_n(h) \to \frac{E(\sqrt{h})}{K(\sqrt{h})}\ uniformly\ on\ [0,1].$$

Here, $c_n(h)$ are defined later.

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Theorem 7.1 ([3]). *Set* $\overline{g}_n(h)$, $\underline{g}_n(h)$ *by*

$$\overline{g}_n(h) := 1 - \sum_{m=0}^{n} 2^{m-1} c_m(h)^2,$$
$$\underline{g}_n(h) := 1 - \sum_{m=0}^{n} 2^{m-1} c_m(h)^2 - 2^{n-1} c_n(h)^2.$$

Then

$$\underline{g}_n(h) \le \frac{E(\sqrt{h})}{K(\sqrt{h})} \le \overline{g}_n(h) \quad \text{on } [0,1] \qquad (n = 0, 1, 2, \cdots).,$$

and

$$\text{left} \le \text{becomes} = \iff h = 0, 1,$$
$$\text{right} \le \text{becomes} = \iff h = 0,$$

Moreover,

$$\underline{g}_n(h),\ \overline{g}_n(h) \to \frac{E(\sqrt{h})}{K(\sqrt{h})} \text{ uniformly on } [0,1].$$

Here, $c_n(h)$ are defined later.

$$\overline{g}_0(h) = 1 - \frac{h}{2},$$

$$\overline{g}_1(h) = \frac{1}{2} - \frac{h}{4} + \frac{(1-h)^{1/2}}{2},$$

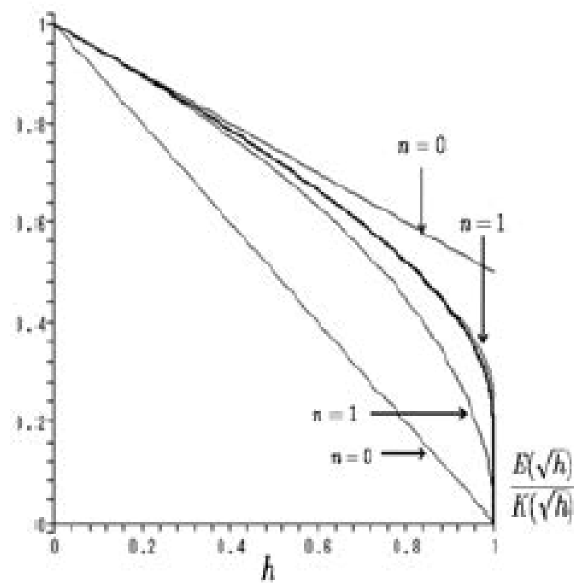
$$\overline{g}_2(h) = \frac{1}{4} - \frac{h}{8} - \frac{(1-h)^{1/2}}{4} + \frac{(1-h)^{1/4}}{2} + \frac{(1-h)^{3/4}}{2},$$

$$\ldots$$

$$\underline{g}_2(h) = (1-h)^{1/4} + (1-h)^{3/4} - (1-h)^{1/2}$$

$$\underline{g}_1(h) = (1-h)^{1/2},$$

$$\underline{g}_0(h) = 1 - h.$$



Let $a, b \quad (a \geq b \geq 0)$ be given. Define $\{a_n\}$, $\{b_n\}$ by

$$a_0 = a, \quad b_0 = b,$$
$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}. \quad (n = 0, 1, 2, \cdots)$$

and $\{c_n\}$ by

$$c_n = \sqrt{a_n^2 - b_n^2} \quad (n = 0, 1, 2, \cdots).$$

We have

$$b = b_0 \leq b_1 \leq \cdots \leq b_n \leq a_n \leq \cdots \leq a_1 \leq a_0 = a,$$

$$\frac{a_n - b_n}{2} = c_{n+1} \leq \frac{c_n}{2} \leq \cdots \leq \frac{c_0}{2^{n+1}} \quad (n = 0, 1, 2, \cdots).$$

Thus $\{a_n\}$, $\{b_n\}$ has a same limit AGM (a, b).

For example,

$$\mathrm{AGM}\left(1, \frac{1}{\sqrt{2}}\right) = 0.8472130847\cdots$$

n	a_n	b_n
0	1.0000000000	0.7071067811
1	0.8535533905	0.8408964152
2	0.8472249029	0.8472012667
3	0.8472130848	0.8472130847
4	0.8472130847	0.8472130847

Theorem(Gauss, 1809)

$$\pi = \frac{2\mathrm{AGM}(1, 1/\sqrt{2})^2}{1 - \sum_{n=0}^{\infty} 2^n c_n^2}$$

upper approximation

$$p_N = \frac{2a_N^2}{1 - \sum_{n=0}^{N} 2^n c_n^2}$$

p_0	4.	00000	00000	00000	00000
p_1	3.	18767	26427	12108	62720
p_2	3.	14168	02932	97653	29391
p_3	3.	14159	26538	95446	49600
p_4	3.	14159	26535	89793	23846
π	3.	14159	26535	89793	23846

Theorem (Gauss, 1808) Let $\mathrm{AGM}(1, \sqrt{1-k^2})$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be defined as above. Then,

$$\frac{1}{K(k)} = \frac{2}{\pi} \cdot \mathrm{AGM}(1, \sqrt{1-k^2}),$$

$$\frac{E(k)}{K(k)} = 1 - \sum_{n=0}^{\infty} 2^{n-1} c_n^2.$$

for $0 \leq k < 1$.

Moreover, the convergence is uniform on [0,1] by

$$AGM(1,0) = 0, \quad c_n = 2^{-n}$$

and

$$\lim_{k \uparrow 1} \frac{1}{K(k)} = 0, \quad \lim_{k \uparrow 1} \frac{E(k)}{K(k)} = 0.$$

To simplify the notation, we put $k := \sqrt{h}$., and use the notations $\mathrm{AGM}(1, \sqrt{1-h})$, $\{a_n(h)\}$, $\{b_n(h)\}$, $\{c_n(h)\}$

Proposition 7.1.

$$\frac{1}{K(\sqrt{h})} = \frac{2}{\pi} \cdot \lim_{n\to\infty} a_n(h) = \frac{2}{\pi} \cdot \lim_{n\to\infty} b_n(h).$$

uniformly on $[0, 1]$.

Proposition 7.2.

$$\frac{E(\sqrt{h})}{K(\sqrt{h})} = 1 - \sum_{n=0}^{\infty} 2^{n-1} c_n(h)^2.$$

uniformly on $[0, 1]$

Comparison function for E/K and $1/K$

Theorem 7.1 ([3]). *Set* $\overline{g}_n(h)$, $\underline{g}_n(h)$ *by*

$$\begin{aligned}\overline{g}_n(h) &:= 1 - \sum_{m=0}^{n} 2^{m-1} c_m(h)^2, \\ \underline{g}_n(h) &:= 1 - \sum_{m=0}^{n} 2^{m-1} c_m(h)^2 - 2^{n-1} c_n(h)^2.\end{aligned}$$

Then

$$\underline{g}_n(h) \leq \frac{E(\sqrt{h})}{K(\sqrt{h})} \leq \overline{g}_n(h) \quad on\ [0,1] \quad (n = 0, 1, 2, \cdots).,$$

and

$$\begin{aligned}&left \leq\ becomes = \iff h = 0, 1, \\ &right \leq\ becomes = \iff h = 0,\end{aligned}$$

Moreover,

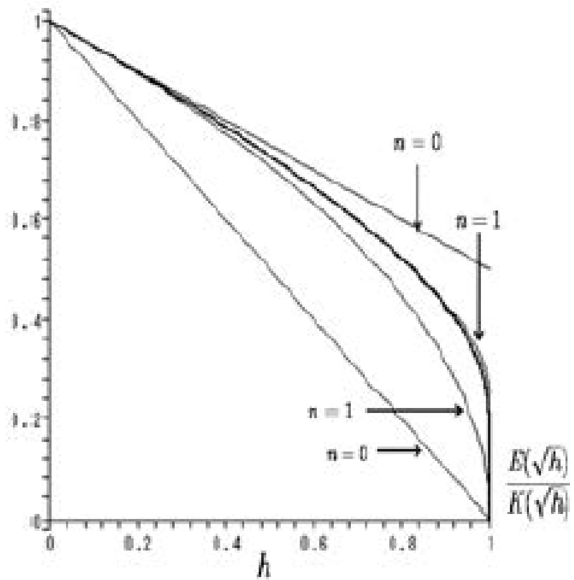
$$\underline{g}_n(h),\ \overline{g}_n(h) \to \frac{E(\sqrt{h})}{K(\sqrt{h})}\ uniformly\ on\ [0,1].$$

Here, $c_n(h)$ are defined later.

$$
\begin{aligned}
\overline{g}_0(h) &= 1 - \frac{h}{2}, \\
\overline{g}_1(h) &= \frac{1}{2} - \frac{h}{4} + \frac{(1-h)^{1/2}}{2}, \\
\overline{g}_2(h) &= \frac{1}{4} - \frac{h}{8} - \frac{(1-h)^{1/2}}{4} + \frac{(1-h)^{1/4}}{2} + \frac{(1-h)^{3/4}}{2},
\end{aligned}
$$

$$\dots$$

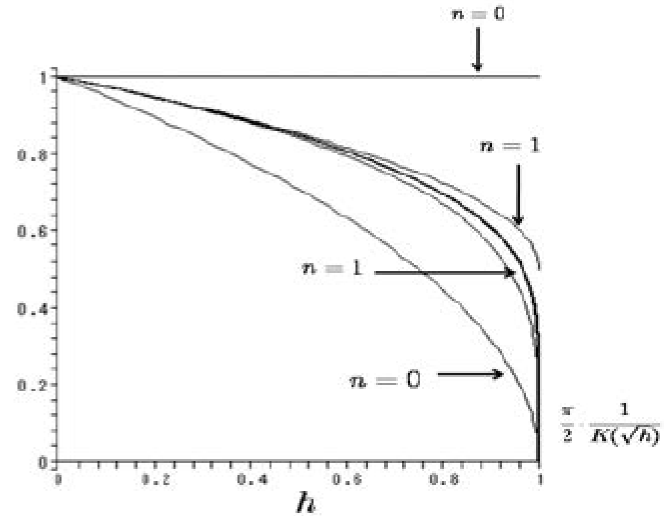
$$
\begin{aligned}
\underline{g}_2(h) &= (1-h)^{1/4} + (1-h)^{3/4} - (1-h)^{1/2} \\
\underline{g}_1(h) &= (1-h)^{1/2}, \\
\underline{g}_0(h) &= 1 - h.
\end{aligned}
$$



Theorem 7.2.

$$\frac{2}{\pi} \cdot b_n(h) \leq \frac{1}{K(\sqrt{h})} \leq \frac{2}{\pi} \cdot a_n(h) \quad (n = 0, 1, 2, \cdots).$$

$$\begin{aligned}
a_0(h) &= 1, \\
a_1(h) &= \frac{1 + (1-h)^{1/2}}{2}, \\
a_2(h) &= \frac{1 + (1-h)^{1/2}}{4} + \frac{(1-h)^{1/4}}{2}, \\
&\cdots \\
b_2(h) &= (1-h)^{1/8} \left(\frac{1 + (1-h)^{1/2}}{2} \right)^{1/2}, \\
b_1(h) &= (1-h)^{1/4}, \\
b_0(h) &= (1-h)^{1/2}.
\end{aligned}$$



Theorem 2 is by Gauss. Theorem 1 is direct application of a theorem by Gauss in 1809 concerning recursive limit of arithmetic algebra mean. We will explain the idea of proofs of them.