Perturbation Theory for Wave Propagation Problems

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1. Introduction

In this talk we consider the classical wave propagation problem in exterior domain.

(1) In the first talk, we explain how linear or non-linear dissipations affect the behavior of energy as time goes to infinity. More precisely, we divide the dissipations which bring about energy decay or non-decay, and in case the energy never decays, we consider the asymptotic behavior in the energy space of the solutions.

There are many works which treat the energy decay or point-wise decay of solutions, see e.g. [2], [4], [8], [10], [12], [14], [17]. On the other hand the energy non-decay and asymptotics of solutions are treated in [5], [6], [8], [10], [12], [13]. We summarize here the results of [5], [8], [13]. Note that the smoothing estimates (linear case) and Strichartz estimates (non-linear case) for free solutions play an essential role to enter into the asymptotoc behavior of perturbed solutions. For these estimates see e.g., [1], [3], [7], [11], [15], [16] and [18].

(2) In the second talk, we treat small perturbation which is linear but non-selfadjoint and depends on space-time. We shall show the uniform boundedness of energy in $t \in \mathbf{R}$ of solutions and apply it to develop the scattering theory, i.e., the existence and unitarity of the Mϕller wave operators.

We summarize here the results of [6], [9].

2. The dissipations which ensure the energy decay

Let Ω be an exterior domain in $\mathbf{R}^n$ $(n \geq 1)$ with smooth compact boundary $\partial\Omega$. We consider in Ω the wave propagation problem

$$\begin{gathered}\partial_t^2 w - \Delta_b w + c(x)w + \beta(x,t,\partial_t w)\partial_t w = 0, \quad (x,t) \in \Omega \times \mathbf{R}_+ \\ w(x,0) = f_1(x), \quad \partial_t w(x,0) = f_2(x), \quad x \in \Omega, \\ w(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbf{R}_+,\end{gathered} \tag{2.1}$$

where

$$\Delta_b u = \nabla_b \cdot \nabla_b u = \sum_{j=1}^n (\partial_j + ib_j(x))^2 u,$$

$$\beta(x,t,\partial_t u) = \tilde{b}(x,t)|\partial_t u|^{\rho-1}$$

with $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$, $\partial_j = \partial/\partial x_j$, $i = \sqrt{-1}$ and $\rho \geq 1$. The coefficients $b_j(x)$, $c(x)$ and $\tilde{b}(x,t)$ are real-valued sufficiently smooth functions. We further require that

$$0 \leq c(x) \leq c_1 \ \text{ for some } c_1 > 0 \ \text{ and } \ \tilde{b}(x,t) \geq 0, \tag{2.2}$$

Thus $\tilde{b}(x,t)|w_t|^{\rho-1}w_t$ represents a friction term.

The operator $L = -\Delta_b + c(x)$ with domain

$$\mathcal{D}(L) = \{u \in L^2(\Omega) \cap H^2_{\rm loc}(\overline{\Omega}) : -\Delta_b u + c(x)u \in L^2(\Omega), u|_{\partial\Omega} = 0\}$$

defines a non-negative selfadjoint operator in $L^2(\Omega)$. Note that $\nabla_b u \in L^2(\Omega)$ if $u \in \mathcal{D}(L)$. Let $H^1_{b,0}$ be the closure in the Dirichlet norm

$$\|u\|_D^2 = \int |\nabla_b u|^2 dx$$

of scalar functions $u \in C_0^\infty(\Omega)$. The Hardy inequality

$$\int \frac{(n-2)^2}{4r^2}|u|^2 dx \leq \int |\nabla_b u|^2 dx$$

holds for each $u \in H^1_{b,0}$.

For solutions $w(x,t)$ of (2.1) we define the energy at time t by

$$\|w(t)\|_E^2 = \frac{1}{2}\int \{|w_t(x,t)|^2 + |\nabla_b w(x,t)|^2 + c(x)|w(x,t)|^2\}dx,$$

In the following we assume that for suitably given initial data $w(0) = \{f_1, f_2\}$, problem (2.1) has a unique global solution with finite energy which also satisfies the energy identity

$$\|w(t)\|_E^2 + \int_0^t \int \beta(x,\tau,w_t(x,\tau))|w_t(x,\tau)|^2 dx d\tau = \|w(0)\|_E^2.$$

To enter into the energy decay problems, we introduce an weighted energy of solutions. Let $\varphi(s)$, $s \geq 0$, be a smooth function satisfying

$$1 \leq \varphi(s) \leq \varphi_0(1+s) \text{ for some } \varphi_0 > 0 \text{ and } \lim_{s\to\infty}\varphi(s) = \infty, \tag{2.3}$$

$$\varphi'(s) \geq 0, \quad \varphi''(s) \leq 0, \quad \varphi'''(s) \geq 0 \text{ and they all are bounded in } s \geq 0, \tag{2.4}$$

With this $\varphi(s)$ an weighted energy of solutions at time t is defined by

$$\|w(t)\|_{E_\varphi}^2 = \frac{1}{2}\int \varphi(r+t)\{|w_t|^2 + |\nabla_b w|^2 + c|w|^2\}dx.$$

We multiply by $\varphi(r+t)\overline{w_t}$ on both sides of the equation and take the real part. Then

$$\frac{1}{2}\partial_t\{\varphi(|w_t|^2 + \nabla_b w|^2 + c|w|^2)\} - \mathrm{Re}\nabla\cdot(\nabla_b w\varphi\overline{w_t})$$
$$-\frac{1}{2}\varphi'(|w_t|^2 + |\nabla_b w|^2) - \frac{1}{2}\varphi' c|w|^2 + \varphi\beta|w_t|^2 = 0.$$

Next multiply by $\varphi'(r+t)\overline{w}$ on both sides of the equation. Then

$$\frac{1}{2}\mathrm{Re}\partial_t\{\varphi'(2w_t\overline{w}\} - \mathrm{Re}\nabla\cdot(\nabla_b w\varphi'\overline{w}) - \varphi'(|w_t|^2 - |\nabla_b w|^2 - c|w|^2)$$

$$-\mathrm{Re}\varphi''(w_t\overline{w} - \tilde{x}\cdot\nabla_b w\overline{w}) + \mathrm{Re}\varphi'\beta w_t\overline{w} = 0.$$

Getting together these equations, we have

$$X_t + \nabla\cdot Y + Z = 0, \quad X_t = \partial_t X, \tag{2.5}$$

where

$$X = \frac{1}{2}\varphi\{|w_t|^2 + |\nabla_b w|^2 + c|w|^2\} + \mathrm{Re}\varphi' w_t\overline{w},$$

$$Y = -\mathrm{Re}\nabla_b w(\varphi\overline{w_t} + \varphi'\overline{w})$$

$$Z = (\varphi\beta - 2\varphi')|w_t|^2 + \frac{1}{2}\varphi'|\tilde{x}w_t + \nabla_b w + \tilde{x}\varphi'^{-1}\varphi'' w|^2$$

$$+\frac{1}{2}\{2\varphi''' - \varphi'^{-1}\varphi''^2 + \varphi' c\}|w|^2 - 2\mathrm{Re}\varphi'' w_t\overline{w} + \mathrm{Re}\varphi'\beta w_t\overline{w}.$$

$$\geq (\varphi\beta - 2\varphi')|w_t|^2 + \mathrm{Re}(\varphi'\beta - 2\varphi'')w_t\overline{w} + \frac{1}{2}\{2\varphi''' - \varphi'^{-1}\varphi''^2 + \varphi' c\}w|^2. \tag{2.6}$$

The case of linear dissipation $\beta w_t = \tilde{b}(x,t)w_t$.

In this case noting

$$(\varphi\beta - 2\varphi')|w_t|^2 = \{\varphi(r+t)\tilde{b}(x,t) - 2\varphi'(r+t))|w_t|^2,$$

$$\mathrm{Re}(\varphi'\beta - 2\varphi')w_t\overline{w} = \frac{1}{2}\partial_t\{(\varphi'\tilde{b} - \varphi')|w|^2)\} - \frac{1}{2}\{(\varphi'\tilde{b})_t - 2\varphi''\}|w|^2,$$

we require

$$b_0(1+r+t)^{-1} \leq \tilde{b}(x,t) \leq b_1(1+r+t) \text{ for some } b_0, b_1 > 0. \tag{2.7}$$

$$\varphi(r+t)\tilde{b}(x,t) - 2\varphi'(r+t) \geq 0, \tag{2.8}$$

$$-\{\varphi'(r+t)\tilde{b}(x,t)\}_t + \varphi'(r+t)c(x) \geq 0, \tag{2.9}$$

$$2\varphi'''(r+t) - \varphi'(r+t)^{-1}\varphi''(r+t)^2 \geq 0 \tag{2.10}$$

for $(x,t) \in \Omega\times(0,\infty)$. Then we have

$$Z \geq \frac{1}{2}\partial_t[(\varphi'\tilde{b} - 2\varphi'')|w|^2],$$

and (2.5) is reduced to

$$\frac{1}{2}\partial_t\left\{\varphi(|w_t|^2 + |\nabla_b w|^2 + c|w|^2) + 2\mathrm{Re}\varphi' w_t\overline{w}\right.$$

$$+(\varphi'\tilde{b}-2\varphi'')|w|^2\Big\} - \mathrm{Re}\nabla\cdot\{\nabla_b w(\varphi\overline{w}_t+\varphi'\overline{w})\}\leq 0$$

Integrate the both sides over $\Omega_R\times(0,t)$. Then

$$\frac{1}{2}\int_{\Omega_R}\Big\{\varphi(|w_t|^2+|\nabla_b w|^2+c|w|^2)+2\mathrm{Re}\varphi' w_t\overline{w}+(\varphi'\tilde{b}-2\varphi'')|w|^2\Big\}dx$$

$$-\frac{1}{2}\int_{\Omega_R}\Big\{\varphi(|f_2|^2+|\nabla_b f_1|^2+c|f_1|^2)+2\mathrm{Re}\varphi' f_2\overline{f}_1+(\varphi'\tilde{b}-2\varphi'')|f_1|^2\Big\}dx$$

$$-\mathrm{Re}\int_0^t\int_{S_R}\tilde{x}\cdot\{\nabla_b w(\varphi\overline{w}_t+\varphi'\overline{w})\}dSd\tau\leq 0.$$

For $w(x,t)$ with finite energy, if it also belongs to $L^2(\Omega)$, it follows that

$$\liminf_{R\to\infty}\int_0^t\int_{S_R}|\tilde{x}\cdot\nabla_b w|\{\varphi|w_t|+\varphi'|w|\}dSd\tau=0$$

since $\varphi(r+\tau)=O(r)$ and $\varphi'(r+\tau)=O(1)$ uniformly in $s\in(0,t)$. Moreover, since

$$\int_{\Omega_R}|\varphi' w_t\overline{w}|dx\leq\frac{1}{2}\int_{\Omega_R}\Big\{\epsilon\varphi|w_t|^2+\epsilon^{-1}\varphi^{-1}\varphi'^2|w|^2\Big\}dx,$$

letting $R\to\infty$ we obtain for any $t>0$ and $0<\epsilon<1$

$$(1-\epsilon)\|w(t)\|_{E_\varphi}^2+\frac{1}{2}\int\{\varphi'\tilde{b}-2\varphi''-\epsilon^{-1}\varphi^{-1}(\varphi')^2\}|w|^2dx$$

$$\leq(1+\epsilon)\|w(0\|_{E_\varphi}^2+\frac{1}{2}\int\{\varphi'\tilde{b}-2\varphi''+\epsilon^{-1}\varphi^{-1}(\varphi')^2\}f_1^2dx.$$

Theorem 2.1 *Assume* (2.2), (2.3) *and* (2.7) – (2.10). *Let* $\{f_1,f_2\}\in\mathcal{D}(L)\times H^1_{b,0}$ *also satisfy*

$$\int\varphi(r)\{|\nabla_b f_1|^2+|f_2|^2\}dx<\infty.$$

Then the solution $w(x,t)$ *of problem* (2.1) *satisfies*

$$\|w(t)\|_{E_\varphi}^2\leq 3\|f\|_{E_\varphi}^2+2\int\{-\varphi''(r)+\varphi'(r)\tilde{b}(x,0)\}|f_1|^2dx. \tag{2.11}$$

Thus, the energy of $w(\cdot,t)$ *decays like*

$$\|w(t)\|_E^2=O(\varphi(t)^{-1})\quad as\quad t\to\infty.$$

Proof We choose $\epsilon=1/2$. Then since

$$\varphi'\tilde{b}-2\varphi^{-1}\varphi'^2=\varphi'\varphi^{-1}(\varphi\tilde{b}-2\varphi')\geq 0$$

by assumption (2.8), assertion (2.11) of the theorem follows. The fact that $w(t)\in C([0,T];L^2)$, $T>0$, is guaranteed by the conditions of the initial data. □

Examples of $\varphi(s)$

(1) $\varphi(s) = (1+s)^\gamma, \quad 0 < \gamma < 1,$

(2) $\varphi(s) = \{\log(e+s)\}^\gamma, \quad 0 < \gamma < 1.$

It is obvious that these examples satisfies (2.4) and (2.5).

In case $b_0(1+r+t)^{-1} \le \tilde{b}(x,t) \le b_1(1+r+t)^{1-\delta}$, $0 < \delta < 1$, we can use (1). By definition (2.8) is verified as

$$2\varphi'''(s) - \varphi(s)^{-1}\varphi''(s)^2 = \gamma(1-\gamma)(1+s)^{\gamma-3} \ge 0.$$

(2.9) holds if $b_0 \ge 2\gamma$ in (2.3). In fact

$$\varphi\tilde{b} - 2\varphi' \ge (1+r+t)^{\gamma-1}\{b_0 - 2\gamma\} \ge 0.$$

(2.10) becomes

$$-\{(1+r+t)^{\gamma-1}\tilde{b}(x,t)\}_t + (1+r+t)^{\gamma-1}c(x) \ge 0,$$

which is satisfied for any $c(x) \ge 0$ if $\{(1+r+t)^{\gamma-1}\tilde{b}(x,t)\}_t \le 0$. On the other hand, if $c(x) = m^2 > 0$ (Klein-Gordon equation case), such a restriction is not necessary if $\tilde{b}_t(x,t) \le m^2 - (1-\gamma)b_1$.

In case $\tilde{b}(x,t) \le b_1(1+r+t)$ it becomes necessary to use (2). (2.8) is given by

$$\begin{aligned} 2\varphi'''(s) - \varphi(s)^{-1}\varphi''(s)^2 &= \gamma\{1 - 2(\gamma-1)\{\log(e+s)\}^{-1} \\ &+(\gamma-2)\{\log(e+s)\}^{-2}\}\{\log(e+s)\}^{\gamma-1}(1+s)^{-3} \\ &\ge \gamma(1-\gamma)\{\log(e+s)\}^{\gamma-1}(e+s)^{-3} \ge 0. \end{aligned}$$

(2.9) also holds if $b_0 \ge 2\gamma$. In fact

$$\varphi\tilde{b} - 2\varphi' \ge \{\log(e+s)\}^\gamma(1+s)^{-1}(b_0 - 2\gamma\{\log(e+s)\}^{-1}) \ge 0.$$

The situation of (2.10) is also similar as above.

The case of non-linear dissipation $\beta w_t = \tilde{b}(x,t)|w_t|^{\rho-1}w_t$.

We require

$$1 < \rho < 1 + \frac{2(1-\delta)}{n} \quad \text{and} \quad b_0(1+r+t)^{-\delta} \le \tilde{b}(x,t) \le b_1 \tag{2.12}$$

for some $0 \le \delta < 1$ and $b_0,$, $b_1 > 0$. Moreover,

$$\tilde{b}(x,t) \text{ is non} - \text{increasing in } t. \tag{2.13}$$

With these conditions we adopt as the weight function the following

$$\varphi(s) = \{\log(e+s)\}^\mu \quad \text{for some} \quad 0 < \mu < \min\Big\{\frac{2}{\rho-1}, \rho\Big\}.$$

Lemma 2.1 *There exist constants $J(\varphi) > 0$ and $K(\varphi, w(0)) > 0$ such that*

$$\int_0^t \int 2\varphi' |w_t|^2 dx d\tau \le J(\varphi)^{(\rho-1)/(\rho+1)} \Big(\int_0^t \int \varphi \tilde{b} |w_t|^{\rho+1} dx d\tau \Big)^{2/(\rho+1)},$$

$$\int_0^t \int |\varphi' \tilde{b} |w_t|^{\rho-1} w_t \overline{w}| dx d\tau \le K(\varphi, w(0))^{1/(\rho+1)} \Big(\int_0^t \int \varphi \tilde{b} |w_t|^{\rho+1} dx d\tau \Big)^{\rho/(\rho+1)}.$$

Proof By the Hölder inequality

$$\int_0^t \int 2\varphi' |w_t|^2 dx d\tau \le J_1(\varphi)^{(\rho-1)/(\rho+1)} \Big(\int_0^t \int \varphi \tilde{b} |w_t|^{\rho+1} dx d\tau \Big)^{2/(\rho+1)},$$

where

$$J_1(\varphi) = \int_0^t \int (\varphi \tilde{b})^{-2/(\rho-1)} \varphi'^{(\rho+1)/(\rho-1)} dx d\tau$$

Note here

$$\begin{aligned} &(\varphi \tilde{b})^{-2/(\rho-1)} \varphi'^{(\rho+1)/(\rho-1)} \\ &\le C \{\log(e + r + t)\}^{\mu - (\rho+1)/(\rho-1)} (e + r + t)^{-1-2(1-\delta)/(\rho-1)}. \end{aligned}$$

Then since $-2(1-\delta)/(\rho-1) \le -n$ and $\mu - (\rho+1)/(\rho-1) < -1$, we have

$$J_1(\varphi) \le C \int_0^t \{\log(e + \tau)\}^{\mu - (\rho+1)/(\rho-1)} d\tau \int_0^\infty (e + r + \tau)^{-2} dr.$$

Hence the first inequality holds with

$$J(\varphi) = C \int_0^\infty \{\log(e + \tau)\}^{\mu - (\rho+1)/(\rho-1)} (e + \tau)^{-1} d\tau < \infty.$$

Next, the Hölder inequality also shows

$$\int_0^t \int |\varphi' \tilde{b} |w_t|^{\rho-1} w_t \overline{w}| dx d\tau \le K_1(\varphi, w(t))^{1/(\rho+1)} \Big(\int_0^t \int \varphi \tilde{b} |w_t|^{\rho+1} dx d\tau \Big)^{\rho/(\rho+1)},$$

where

$$\begin{aligned} K_1(\varphi, w(t)) &= \int_0^t \int \varphi^{-\rho} b (\varphi' |w|)^{\rho+1} dx d\tau \\ &\le \int_0^t \varphi(\tau)^{-\rho} \varphi'(\tau)^{\rho+1} d\tau \int \tilde{b} |w|^{\rho+1} dx. \end{aligned}$$

Since $w(\tau) = \int_0^\tau w_t(s) ds + w(0)$, noting $\tilde{b}(x,t)$ being non-increasing in t, we have

$$\begin{aligned} \int \tilde{b} |w(\tau)|^{\rho+1} dx &\le C_1 \int \Big\{ \tau^\rho \int_0^\tau \tilde{b} |w_t(s)|^{\rho+1} ds + |w(0)|^{\rho+1} \Big\} dx \\ &\le C_1 (a + \tau)^\rho \left\{ \|w(0)\|_E^2 + \|w(0)\|_{L^{\rho+1}}^{\rho+1} \right\}. \end{aligned}$$

Thus, it follows that

$$K_1(\varphi, w(t)) \leq C \int_0^\infty \{\log(e+\tau)\}^{\mu-\rho-1}(e+\tau)^{-1} d\tau < \infty.$$

and the second inequality holds if we choose

$$K(\varphi, w(0)) = C \int_0^\infty \{\log(e+\tau)^{\mu-\rho-1}(e+\tau)^{-1} d\tau \left\{ \|w(0)\|_E^2 + \|w(0)\|_{H^1}^{\rho+1} \right\}.$$

□

Theorem 2.2 *Assume* (2.2), (2.12) *and* (2.13). *If we choose*

$$0 < \mu < \min\left\{ \frac{2}{\rho-1}, \rho, 1 \right\}$$

then the solution $w(x,t)$ *of* (2.1) *satisfies*

$$\|w(t)\|_{E_\varphi}^2 \leq C\{\||f\|_{E_\varphi}^2 + J(\varphi) + K(\varphi, f)\} + \int \{-\varphi''(r)|f_1|^2 + \varphi'(r)|f_1 f_2|\} dx$$

Thus, the energy of $w(\cdot, t)$ *decays like*

$$\|w(t)\|_E^2 = O(\{\log(e+t)\}^{-\mu})) \quad as \quad t \to \infty.$$

Proof Integrating (2.6) and using the inequalities of Lemma 2.1, we have

$$\int_0^t \int Z dx d\tau \geq \int_0^t \int \varphi \tilde{b} |w_t|^{\rho+1} dx d\tau - J(\varphi)^{(\rho-1)/(\rho+1)} \left(\int_0^t \int \varphi \tilde{b} |w_t|^{\rho+1} dx d\tau \right)^{2/(\rho+1)}$$

$$-K(\varphi, f)^{1/(\rho+1)} \left(\int_0^t \int \varphi \tilde{b} |w_t|^{\rho+1} dx d\tau \right)^{\rho/(\rho+1)}$$

$$\geq (1 - 2\epsilon_1) \int_0^t \int \varphi \tilde{b} |w_t|^{\rho+1} dx d\tau - C(\epsilon_1) \{J(\varphi) + K(\varphi, f)\}$$

for any $\epsilon_1 > 0$, where in the last inequality we have used the Young inequality. Hence it follows that

$$\frac{1}{2} \int \left\{ \varphi(|w_t|^2 + |\nabla_b w|^2 + c|w|^2) + 2\mathrm{Re}\varphi' w_t \overline{w} - 2\varphi''|w|^2 \right\} dx$$

$$-\frac{1}{2} \int \left\{ \varphi(|f_2|^2 + |\nabla_b f_1|^2 + c|f_1|^2) + 2\mathrm{Re}\varphi' f_2 \overline{f}_1 - 2\varphi'')|f_1|^2 \right\} dx$$

$$+(1 - 2\epsilon_1) \int_0^t \int \varphi \tilde{b} |w_t|^{\rho+1} dx d\tau - C(\epsilon_1) \{J(\varphi) + K(\varphi, f)\} \leq 0,$$

which implies the assertion of the theorem if we note

$$-\varphi''(r) - \varphi^{-1}(r)\varphi'(r)^2 \geq 0 \quad \text{when } 0 < \mu < 1.$$

□

3. Energy non-decay and asymptotics for linear dissipations

In this section we require contrary to (2.7) the following conditions on $\tilde{b}(x,t)$:

$$0 \le \tilde{b}(x,t) \le b_0(1+r)^{-1-\delta} \quad \text{for some } 0 < \delta < 1. \tag{3.1}$$

First we consider the simplest case in $\mathbf{R}^n$:

$$\begin{aligned} &\partial_{tt}w - \Delta w + \tilde{b}(x,t)\partial_t w = 0, \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}_+, \\ &w(x,0) = f_1(x), \quad \partial_t w(x,0) = f_2(x), \quad x \in \mathbf{R}^n. \end{aligned} \tag{3.2}$$

In this case the energy of solutions at time t is given by

$$\|w(t)\|_E^2 = \frac{1}{2}\int \{|w_t(x,t)|^2 + |\nabla w(x,t)|^2\}dx,$$

and the following identity holds

$$\|w(t)\|_E^2 + \int_0^t \int \tilde{b}(x,t)|w_t(x,\tau)|^2 dx d\tau = \|w(0)\|_E^2. \tag{3.3}$$

The following lemma is well known (see, e.g., Kato-Yajima [3], Yafaev [18]).

Lemma 3.2 *For $\kappa \in \mathbf{C}_+$ let $R_0(\kappa^2) = (-\Delta - \kappa^2)^{-1}$ be the resolvent of the operator $-\Delta$. Then $u = R_0(\kappa^2)f$ satisfies for $1/2 < \alpha < 1$*

$$\int (1+r)^{-2\alpha}\{|\nabla u|^2 + |\kappa u|^2\}dx \le C\int (1+r)^{2\alpha}|f|^2 dx$$

where $C > 0$ is independent of f and κ.

For solution w of (3.2) we put $u = \{w, w_t\}$. Then u satisfies the equation

$$iu_t = Mu + V(t)u, \quad u(0) = f = \{f_1, f_2\}, \tag{3.4}$$

where

$$M = i\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \quad \text{and} \quad V(t) = i\begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}(x,t) \end{pmatrix},$$

M is selfadjoint in the energy space $\mathcal{H}_E$. So, it defines a unitary group $\{e^{-itM}; t \in \mathbf{R}\}$. On the other hand, let $\{U(t,s); t \ge s\}$ be the family of contraction operators which represent the solution at time s to that of time t.

The integral expression of (3.4) gives

$$U(t,0)f = e^{-itM}f + \int_0^t e^{-i(t-\tau)M}V(\tau)U(\tau,0)f d\tau.$$

Then we have

$$(U(t,s)f, e^{-i(t-s)M}g)_E = (f,g)_E + \int_s^t (V(\tau)U(\tau,s)f, e^{i(\tau-s)M}g)_E d\tau. \tag{3.5}$$

Theorem 3.3 *(i) Under assumtion* (3.1) *the wave operator*

$$Z(s) = s - \lim_{t\to\infty} e^{i(t-s)M}U(t,s)$$

exists for any $s \geq 0$.

(ii) $Z(s)$ *is not identically vanishing in* $\mathcal{H}_E$.

Proof (i) We put $\mathcal{R}_0(\kappa) = (\Lambda_0 - \kappa)^{-1}$ for $\kappa \in \mathbf{C}\backslash\mathbf{R}$, and

$$A = \begin{pmatrix} 0 & 0 \\ 0 & a(x) \end{pmatrix}, \quad a(x) = \sqrt{b_0}(1+r)^{-(1+\delta)/2},$$

Then since

$$\|A\mathcal{R}_0(\kappa)Af\|_E = \| - ia\kappa R_0(\kappa^2)af_2\|,$$

it follows from Lemma 3.2 with $\alpha = (1+\delta)/2$ that

$$\|A\mathcal{R}_0(\kappa)Af\|_E \leq C\|f_2\| \leq C\|f\|_E,$$

which implies the smoothing estimate

$$\int_0^\infty \|Ae^{-itM}f\|_E^2 dt \leq C\|f\|_E^2. \tag{3.6}$$

Now we return to (3.5). Then since $\sqrt{V(t)} \leq A$ it follows that

$$(U(t,s)f, e^{-i(t-s)M}g)_E - (U(t_1,s)f, e^{-i(t_1-s)M}g)_E = \int_{t_1}^t (V(\tau)U(\tau,s)f, e^{i(\tau-s)M}g)_E d\tau$$

$$\leq \Big(\int_{t_1}^t \|\sqrt{V(\tau)}U(\tau,s)f\|_E^2 d\tau\Big)^{1/2}\Big(\int_{t_1}^t \|Ae^{-i(\tau-s)M}g\|_E^2 d\tau\Big)^{1/2}. \tag{3.7}$$

This and (3.3) and (3.6) show the assertion.

(ii) / To show the existence of $f \in \mathcal{H}_E$ such that $Z(0)f \neq 0$. We assume contrary that $\|U(t,0)f\|_E \to 0$ as $t \to \infty$. Then we have from (3.3)

$$\|f\|_E^2 = \int_0^\infty \|\sqrt{V(\tau)}U(\tau,0)f\|^2 d\tau.$$

Further, from (3.5) and (3.7)

$$\|f\|_E^2 \leq \Big(\int_0^\infty \|\sqrt{V(\tau)}U(\tau,0)f\|_E^2 d\tau\Big)^{1/2}\Big(\int_0^\infty \|Ae^{-i(\tau)M}g\|_E^2 d\tau\Big)^{1/2}.$$

Hence it follows that

$$\|f\|_E^2 \leq \int_0^\infty \|Ae^{-i\tau M}f\|_E^2 d\tau.$$

Put here $f = U(s)g$ with $\|g\|_E = 1$. Then

$$1 = \|e^{-isM}g\|_E^2 \leq \int_s^\infty \|Ae^{-i\tau M}g\|_E^2 d\tau \to 0 \ \text{ as } s \to \infty.$$

this is a contradiction, and (ii) is proved. □

If $n \geq 3$ and if Ω is the exterior of some star-shaped obstacle, the above results can be generalized to problem (2.1). In this case, we can prove the following lemma corresponding to Lemma 3.1 (see Mochizuki [//]).

Lemma 3.3 *Asume $c(x) = c_0(x) + c_1(x)$ and*

$$\{|\nabla \times b(x)|^2 + |c_0(x)|^2\}^{1/2} \leq -\epsilon(1+r)^{-2-\delta_1} \quad \text{for some } 0 < \delta_1 < 1,$$

$$c_1(x) \geq 0, \quad \partial_r\{rc_1(x)\} \leq 0, \quad c_1(x) = o(r^{-1}) \ (r \to \infty),$$

where $\epsilon > 0$ is chosen sufficiently small. Under these requirement let $R(\kappa^2) = (L - \kappa^2)^{-1}$. Then $u = R(\kappa^2)f$ satisfies for $1/2 < \alpha \leq (1+\delta_1)/2$

$$\int (1+r)^{-2\alpha}\{|\nabla_b u|^2 + |\kappa u|^2\}dx \leq C \int (1+r)^{2\alpha}|f|^2 dx,$$

where $C > 0$ is independent of f and κ.

Let $u = \{w, w_t\}$. then (2.1) is rewritten in the form

$$\partial_t u = Mu + V(t)u, \quad u(0) = f = \{f_1, f_2\},$$

where

$$M = \begin{pmatrix} 0 & 1 \\ \Delta_b - c & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}(x,t) \end{pmatrix}$$

The energy space $\mathcal{H}_E$ with norm

$$\|f\|_E^2 = \frac{1}{2}\int_\Omega \{|f_2|^2 + |\nabla_b f_1|^2 + c|f_1|^2\}dx$$

In $\mathcal{H}_E$ the operator M forms a selfadjoint operator with domain

$$\mathcal{D}(\Lambda) = \{f_1 \in H^1_{b,0}; \Delta_b f_1 \in L^2\} \times \{f_2 \in H^1_{b,0} \cap L^2\}.$$

Let $\{e^{-itM}; t \in \mathbf{R}\}$ be the unitary family of free evolution. Let $\{U(t,s); t \geq s \geq 0\}$ be the perturbed evolution. Then we see that

$$Z(s) = s - \lim_{t\to\infty} e^{i(t-s)M}U(t,s)$$

exists for any $s \geq 0$ and defines a not identically vanishing operator.

4. The case of nonlinear dissipations

We consider the dissipative wave equation

$$w_{tt} - \Delta w + \beta(x,t,w_t)w_t = 0, \quad \beta(x,t,w_t) = \tilde{b}(x,t)|w_t|^{\rho-1}, \tag{4.1}$$

in $x \in \mathbf{R}^n$ and $t > 0$, with initial conditions

$$\{w(0), w_t(0)\} = \{f_1, f_2\} \in H^{2,2} \times (H^{1,2} \cap L^{2\rho}). \tag{4.2}$$

Here $\rho \geq 1$ and the coefficient $b(x,t)$ is required to satisfy

$$C^{-1}\{|\tilde{b}_t(x,t)| + |\nabla\tilde{b}(x,t)\} \leq \tilde{b}(x,t) \leq b_1 < \infty, \quad C,\ b_1 > 0. \tag{4.3}$$

In the following let $H^{r,q}$ ($r \in \mathbf{R}, q \geq 1$) be the Sobolev space and $\dot{H}^{r,q}$ ($r > -n, q \geq 1$) be the homogeneous Sobolev space, respectively, given by

$$H^{r,q} = \{u; \|u\|_{H^{r,q}} = \|\mathcal{F}^{-1}(<\xi>^r \hat{u}(\xi))\|_{L^q} < \infty\},$$

$$\dot{H}^{r,q} = \{u; \|u\|_{\dot{H}^{r,q}} = \|\mathcal{F}^{-1}(|\xi|^r \hat{u}(\xi))\|_{L^q} < \infty\}.$$

Here $<\xi>= \sqrt{1+|\xi|^2}$.

As is shown in Motai [12], a unique strong solution exists and it satisfies

(a) $$w(t) \in C([0,\infty); H^{1,2} \times L^2),$$

(b) $$\|w(t)\|_E^2 + 2\int_0^t \int_{\mathbf{R}^n} \beta(x,s,w_t))|w_t(x,s)|^2 dxds = \|w(0)\|_E^2,$$

(c) $$w_{tt},\ \nabla w_t,\ \Delta w,\ \tilde{b}(x,t,w_t) \in L^\infty((0,\infty); L^2).$$

Here

$$\|w(t)\|_E^2 = \frac{1}{2}\int_{\mathbf{R}^n} \{|w_t(x,t)|^2 + |\nabla w(x,t)|^2\} dx.$$

Since $\tilde{b}(x,t) \geq 0$, (b) shows that the energy of solution decreases with t, and a question arises: whether it decays or not as $t \to \infty$?

As an answer we have the following:

Theorem 4.4 (i) *Other than* (4.3) *if*

$$\tilde{b}(x,t) \leq b_0(1+|x|)^{-\delta}, \quad \rho > 1 + \frac{2(1-\delta)}{n-1},$$

then the energy of solutions of (4.1), (4.2) *does not in general decay as* $t \to \infty$.

(ii) Other than (4.3) *if*

$$0 \le \tilde{b}(x,t) \le b_1(1+|x|)^{-\delta}, \quad 1+\frac{4(1-\delta)}{n-1} < \rho < 1+\frac{6}{n-2}. \tag{4.4}$$

Then for each solution $w(t)$ *of* (4.1), (4.2) *there exists* $f_0^+ = \{f_{01}^+, f_{02}^+\}$ *in the energy space* $\mathcal{H}_E$ *such that*

$$\lim_{t\to\infty} \|e^{it\Lambda_0}w(t) - f_0^+\|_E = 0.$$

Assertion (i) is proved similar to Theorem 3.3 (ii) if we use

Lemma 4.4 *If* f *satisfies*

$$\|f\|_{\Gamma,m+2} = \sum_{|\alpha|,|\beta|\le m+2} \|x^\alpha \nabla^\beta f_1\| + \sum_{|\alpha|,|\beta|\le m+1} \|x^\alpha \nabla^\beta f_2\| < \infty$$

with $m = [n/2]$*, then we have*

$$|[e^{-it\Lambda_0}f]_2(x)| \le C(1+r+t)^{-(n-1)/2}(1+|r-t|^{-1/2}\|f\|_{\Gamma,m+2}.$$

Assertion (ii) is based on, as in the linear case (see (3.7)), the equation

$$(w(t), e^{-itM}f_0)_E - (w(s), e^{-isM}f_0)_E = -\int_s^t (\beta(\cdot,\tau,w_t)w_t, [e^{-i\tau M}f_0]_2)d\tau.$$

Lemma 4.5 *Under* (4.3) *and* (4.4) *we have*

$$\beta(x,t,w_t)w_t = \tilde{b}(x,t)|w_t|^{\rho-1}w_t \in L^{(\rho+1)/\rho}((0,\infty); H^{1,(\rho+1)/\rho}_{\delta/(\rho+1)}),$$

where for $p \ge 1$ *and* $\gamma \in \mathbf{R}$

$$\|u\|^p_{H^{k,p}_\gamma} = \sum_{J=0}^k \int |\nabla^j\{(1+r)^\gamma u(x)\}|^p dx \quad k = 0,1,2\cdots.$$

Proof The energy equation (b) shows

$$\tilde{b}(x,t)|w_t|^\rho = \tilde{b}^{1/(\rho+1)}[\tilde{b}|w_t|^{\rho+1}]^{\rho/(\rho+1)} \in L^{(\rho+1)/\rho}((0,\infty); L^{(\rho+1)/\rho}_{\delta/(\rho+1)})$$

since we have $\tilde{b}^{1/(\rho+1)} \le [b_1(1+r)^{-\delta}]^{1/(\rho+1)}$. Noting (4.3) we have a similar energy equation for ∇w. Then the results are summarized as in above lemma. □

With this lemma we can have

$$\begin{aligned}&|(w(t), e^{-it\Lambda_0}f_0)_E - (w(s), e^{-is\Lambda_0}f_0)_E|\\ &\quad \le \|\beta(\cdot,\tau,w_t)w_t\|_{L^{(\rho+1)/\rho}(s,t;Y')}\|[e^{-i\tau\Lambda_0}f_0]_2\|_{L^{\rho+1}(s,t;Y)},\end{aligned}$$

where $Y' = H^{1,(\rho+1)/\rho}_{\delta/(\rho+1)}$. So, our problems become to show that for suitable space $Z \subset \mathcal{H}_E$

$$(*) \qquad \|[e^{-i\tau\Lambda_0} f_0]_2\|_{L^{\rho+1}((0,\infty);Y)} \leq C\|f_0\|_Z.$$

$\{e^{it\Lambda_0}w(t)\}$ being bounded in $\mathcal{H}_E$, it weekly converges in $\mathcal{H}_E$. Let f_0^+ denote the limit. Then letting $t \to \infty$ in (4.//) and replacing s by t, we obtain

$$|(e^{it\Lambda_0}w(t) - f_0^+, f_0)_E| \leq$$

$$\leq C\|b(\cdot,\cdot,w_t)\|_{L^{(\rho+1)/\rho}((t,\infty);Y')}\|f_0\|_Z,$$

from which it follows that

$$\|e^{it\Lambda_0}w(t) - f_0^+\|_{Z'} \leq C\|b(\cdot,\cdot,w_t)\|_{L^{(\rho+1)/\rho}((t,\infty);Y')} \to 0,$$

as $t \to \infty$. So, if we have the embedding

$$(**) \qquad Z' \hookrightarrow \mathcal{H}'_E,$$

then this proves the desired assertion (ii).

To verify $(*)$ and $(**)$ we use a weighted Strichartz estimate obtained by combined with the usual Strichatz and smoothing estimates.

We put $K = \sqrt{-\Delta}$. Then

$$\|K^r u\|_{L^q} = \|u\|_{\dot{H}^{r,q}}.$$

The following is due to Strichartz [15]: Let p and α satisfy

$$1 - \frac{2\alpha}{n+1} \leq \frac{2}{p} \leq 1 - \frac{\alpha}{n}, \quad 0 < \alpha < n, \tag{4.5}$$

Then we have for $u \in C_0^\infty(\mathbf{R}^n)$ and $d_1 = n\Big(\frac{1}{p'} - \frac{1}{p} - \frac{\alpha}{n}\Big)$

$$\|K^{-\alpha}e^{itK}u\|_{L^p} \leq C|t|^{-d_1}\|u\|_{L^{p'}}. \tag{4.6}$$

This is used to show the following

Lemma 4.6 *If we restrict p to satisfy $0 < d_1 < 1$. Then putting $\frac{1}{r'} = 1 - \frac{d_1}{2}$, we obtain for $h(t) \in C_0(\mathbf{R}; L^{p'})$*

$$\Big\{\int_{-\infty}^{\infty} \|e^{iKt}u\|_{L^p}^r dt\Big\}^{1/r} \leq C\|u\|_{\dot{H}^{\alpha/2,2}}. \tag{4.7}$$

Proof By the Fubini theorem it follows from (4.6) that

$$\left\|\int_{-\infty}^{\infty} K^{-\alpha}e^{-itK}h(t)dt\right\|^2_{\dot{H}^{\alpha/2,2}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(K^{-\alpha}e^{-i(t-s)K}h(t), h(s)\Big)\, dtds$$

$$\le \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} C|t-s|^{-d_1}\|h(t)\|_{L^{p'}}dt\Big]\|h(s)\|_{L^{p'}}ds.$$

Since $0 < d_1 < 1$, choosing q, $r > 1$ to satisfy $\dfrac{1}{r} = \dfrac{1}{q} - (1-d_1)$, we can apply the Hardy-Littlewood theorem to obtain

$$\le C\|h\|_{L^q(\mathbf{R};L^{p'})}\|h\|_{L^{r'}(\mathbf{R};L^{p'})}.$$

Thus, the inequality

$$\left\|\int_{-\infty}^{\infty} K^{-\alpha}e^{-itK}h(t)dt\right\|_{\dot{H}^{\alpha/2,2}} \le C\|h\|_{L^{r'}(\mathbf{R};L^{p'})}$$

holds if we choose $\dfrac{1}{q} = \dfrac{1}{r'}$, i.e., $\dfrac{1}{r'} = 1 - \dfrac{d_1}{2}$.

Inequality (4.7) is a result of the duality argument. □

The following is a well known smoothing estimate (see e.g., Hoshiro [1], Sugimoto [16]):

Lemma 4.7 *Let $n \ge 2$ and $\dfrac{1}{2} < \beta < \dfrac{n}{2}$. Then we have*

$$\|e^{itK}u\|_{L^2(\mathbf{R};L^2_{-\beta})} = \left\{\int_{-\infty}^{\infty}\|e^{iKt}u\|^2_{L^2_{-\beta}}dt\right\}^{1/2} \le C\|u\|_{\dot{H}^{\beta-1/2,2}}, \tag{4.8}$$

where for $p \ge 1$ and $\gamma \in \mathbf{R}$

$$\|v\|^p_{L^p_\gamma} = \int_{\mathbf{R}^n}|<x>^\gamma v(x)|^p dx.$$

Put $r = p$ in (4.7), and interpolate this and (4.8). Then since $\dfrac{1}{p} = \dfrac{n-\alpha}{2(n+1)}$, we have the following

Lemma 4.8 *The following inequality holds for $0 \le \theta \le 1$.*

$$\left\{\int_{-\infty}^{\infty}\|e^{iKt}u\|^q_{L^q_{-\mu}}dt\right\}^{1/q} \le C\|u\|_{\dot{H}^{k,2}}.$$

Here q, μ and k are given by

$$(***)\quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2} = \frac{n-\alpha+(1+\alpha)\theta}{2(n+1)},\ \mu = \beta\theta,\ k = \frac{\alpha(1-\theta)}{2} + \Big(\beta - \frac{1}{2}\Big)\theta.$$

Note that for p given above the condition on α becomes

$$1 \le \alpha < n. \tag{4.9}$$

Now, we choose $q = \rho + 1$ and $0 \le \mu \le \dfrac{\delta}{\rho+1}$ in the above estimate. Then noting

$$[e^{-it\Lambda_0} f_0]_2 = w_{0t}(t) = -K \sin(Kt) f_{01} + \cos(Kt) f_{02}$$

and $H^{-1,\rho+1}_{-\mu} \hookrightarrow Y$, we obtain from Lemma 4.7

$$\|[e^{-i\tau\Lambda_0} f_0]_2\|_{L^{\rho+1}((0,\infty);Y)} \le C\|f_0\|_{H^{-1,k+1,2}\times H^{-1,k,2}}. \tag{4.14}$$

Here $H^{a,b,2} = \{u; \|u\|_{H^{a,b,2}} = \|K^b u\|_{H^{a,2}} < \infty\}$.

This proves $(*)$ with $Z = H^{-1,k+1,2} \times H^{-1,k,2}$. Moreover, $(**)$ is also proved if we choose $0 < k \le 1$. In fact, this leads us to the embedding

$$H^{1,1-k,2} \times H^{1,-k,2} \hookrightarrow H^{1-k,1,2} \times H^{1-k,2} \hookrightarrow \dot{H}^{1,2} \times L^2.$$

Now the proof of (ii) becomes complete if we can verify the following

Lemma 4.9 *When ρ satisfies* (4.4)*, the triplet q, μ, k in the above lemma can be chosen to satisfy*

$$\frac{1}{q} = \frac{1}{\rho+1}, \ \mu \le \frac{\delta}{\rho+1}, \ 0 < k \le 1.$$

Proof The above condition on μ is rewritten as $0 \le \theta \le \dfrac{\delta}{(\rho+1)\beta}$. The condition of q is

$$\frac{n-\alpha+(1+\alpha)\theta}{2(n+1)} = \frac{1}{\rho+1}.$$

So, for fixed α and β, ρ takes the maximum when $\theta = 0$

$$\rho = 1 + \frac{2(1+\alpha)}{n-\alpha} \tag{4.10}$$

and the minimum when $\theta = \dfrac{\delta}{(\rho+1)\beta}$

$$\rho = 1 + \frac{2(1+\alpha)(1-\delta)}{n-\alpha} + \frac{(1+\alpha)\delta(2\beta-1)}{(n-\alpha)\beta}. \tag{4.11}$$

Since $k = \alpha/2$ when $\theta = 0$, letting $\alpha \uparrow 2$, we see that the upper-limit of ρ becomes $1 + \dfrac{6}{n-2}$. This gives the maximum of ρ since we can choose $\alpha = 2$. Next consider

(4.11). The right is monotone increasing with both α and β. Thus, letting $\alpha \downarrow 1$ and $\beta \downarrow 1/2$, we obtain the lower-bound $\rho = 1 + \dfrac{4(1-\delta)}{n-1}$, at the same time $k \downarrow 0$. These show the assertion of the lemma. □

Remark When ρ sartisfies

$$1 + \frac{2(1-\delta)}{n-1} < \rho \le 1 + \frac{4(1-\delta)}{n-1} \tag{4.16}$$

we can prove the following: If (4.16) is satisified, there exist $w_0^+(0) \in \mathcal{H}_E$ and $p > 2$ such that

$$\lim_{t\to\infty} \|U_0(-t)u(t) - w_0^+(0)\|_{\dot{H}^{1,p}\times L^p} = 0.$$

5. Time dependent small perturbations of the Klein-Gordon eqution

Let $n \ge 2$ and let Ω be an exterior domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ which is star-shaped with respect to the origin 0 (the case $\Omega = \mathbf{R}^n$ is not excluded when $n \ge 3$). We consider in Ω

$$\partial_t^2 w - \Delta_b w + m^2 w + b_0(x,t)\partial_t w + c(x,t)w = 0, \tag{5.1}$$

where $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, Δ_b is the magnetic Laplacian

$$\Delta_b = \nabla_b \cdot \nabla_b = \sum_{j=1}^{n} (\partial_j + i b_j(x))^2$$

with $\partial_j = \partial/\partial x_j$, m is a positive constant, $b_j(x)$ $(j = 1, \cdots, n)$ are real-valued smooth functions of $x \in \mathbf{R}^n$, and $c(x,t)$, $b_0(x,t)$ are complex-valued continuous functions of $(x,t) \in \mathbf{R}^n \times \mathbf{R}$. For solutions $u = u(x,t)$ and $w = w(x,t)$ we require the zero Dirichlet conditions

$$w(x,t)|_{\partial\Omega} = 0 \tag{5.2}$$

on the boundary $\partial\Omega$. $b(x) = (b_1(x), \cdots, b_n(x))$ represents a magnetic potential. Thus, the magnetic field is defined by its rotation $\nabla \times b(x) = \{\partial_j b_k(x) - \partial_k b_j(x)\}_{j<k}$. We require

$(A1)$ $$|\nabla \times b(x)| \le \epsilon_0 (1 + [r])^{-2}, \quad r = |x|,$$

Here ϵ_0 is a small positive constant and

$$[r] = \begin{cases} r, & \text{when } n \ge 3 \\ r(1 + \log r/r_0), & \text{when } n = 2 \end{cases}$$

for a fixed $r_0 > 0$ satisfying $\partial\Omega \subset \{x; |x| > r_0\}$. As for the coefficients of the perturbation terms, we require the following:

$(A2)$ $$|b_0(x,t)|,\ |c(x,t)| \le \eta(t) + \epsilon_1 (1 + [r])^{-2},$$

where $\eta(t)$ is a positive L^1-function of $t \in \mathbf{R}$ and ϵ_1 is a small positive constant.

Equation (5.1) is rewritten in the system to the pair $\{w, w_t\}$ ($w_t = \partial_t w$):

$$\partial_t \begin{pmatrix} w \\ w_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta_b - m^2 & 0 \end{pmatrix} \begin{pmatrix} w \\ w_t \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ c(x,t) & b_0(x,t) \end{pmatrix} \begin{pmatrix} w \\ w_t \end{pmatrix}.$$

It is considered in the energy space $\mathcal{H}_E = H^1_{b,0} \times L^2$, where $H^1_{b,0} = H^1_{b,0}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with norm

$$\|f\|^2_{H^1_{b,0}} = \int_\Omega \{|\nabla_b f(x)|^2 + |f(x)|^2\} dx.$$

Thus, the inner product and norm of $\mathcal{H}_E$ are given for $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathcal{H}_E$ by

$$(f, g)_{\mathcal{H}_E} = \frac{1}{2} \int_\Omega \left\{ \nabla_b f_1(x) \overline{\nabla_b g_1(x)} + m^2 f_1(x) \overline{g_1(x)} + f_2(x) \overline{g_2(x)} \right\} dx \tag{5.3}$$

and $\|f\|_{\mathcal{H}_E} = \sqrt{(f,f)_{\mathcal{H}_E}}$, respectively. We define the operator M in $\mathcal{H}_E$ by

$$M = \begin{pmatrix} 0 & i \\ i(\Delta_b - m^2) & 0 \end{pmatrix},$$

with domain

$$\mathcal{D}(M) = \left\{ f = \{f_1, f_2\} \in [H^2_{\text{loc}} \cap H^1_{b,0}] \times H^1_{b,0}; \Delta_b f_1 \in L^2 \right\}. \tag{5.4}$$

Then it forms a selfadjoint operator in $\mathcal{H}_E$, and (5.1) with boundary condition (5.2) is represented as

$$i\partial_t u = Mu + V(t)u \quad \text{in } \mathcal{H}_E, \tag{5.5}$$

where $u = \{w, w_t\}$ and

$$V(t)u = \begin{pmatrix} 0 & 0 \\ -ic(x,t) & -ib_0(x,t) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Moreover, by use of the unitary group of operators $\{e^{-itM}; t \in \mathbf{R}\}$ in $\mathcal{H}_E$, (5.5) with initial data $u = \{w(0), w_t(0)\} = \{f_1, f_2\} \in \mathcal{H}_E$ reduces to the integral equation

$$u(t) = e^{-itM} f - i \int_0^t e^{-i(t-\tau)M} V(\tau) u(\tau) d\tau. \tag{5.6}$$

We define the weighted energy space

$$X_E = \Big\{ f(x) = \{f_1(x), f_2(x)\};$$

$$\|f\|^2_{X_E} = \frac{1}{2} \int_\Omega (1 + [r])^{-2} \{|\nabla_b f_1|^2 + m^2 |f_1|^2 + |f_2|^2\} dx < \infty \Big\}. \tag{5.7}$$

For an interval $I \subset \mathbf{R}$ and a Banach space W, we denote by $L^2(I; W)$, the space of all W-valued functions $h(t)$ satisfying

$$\|h\|_{L^2(I;W)} = \left(\int_I \|h(t)\|_W^2 dt \right)^{1/2} < \infty.$$

Similarly, $C(I; W)$ denotes the space of all W-valued continuous functions of $t \in I$. Further, we denote by $\mathcal{B}(W)$ the space of bounded operators on W.

Now, the main results of this paper are summarized in the following theorems.

Theorem 5.5 *For $\zeta \in \mathbf{C} \backslash \mathbf{R}$ put $\mathcal{R}(\zeta) = (M - \zeta)^{-1}$. If ϵ_0 in (A1) is chosen small enough, then there exists $C_0 > 0$ such that*

$$\sup_{\zeta \in \mathbf{C} \backslash \mathbf{R}} \|\mathcal{R}(\zeta) f\|_{X_E} \leq C_0 \|f\|_{X'_E},$$

for each $f \in X'_E$, where X' is the dual space of X_E with respect to $\mathcal{H}_E$.

Theorem 5.6 *Assume (A1) and (A2) with small ϵ_0 and ϵ_1. Then for each $f \in \mathcal{H}$ there exists a unique solution $u(t) \in C(\mathbf{R}; \mathcal{H}_E)$ to the integral equation (11). Let $U(t, s)$, s, $t \in \mathbf{R}_\pm$, be the evolution operator which maps $u(s)$ to $u(t) = U(t, s)u(s)$. Then there exists $C_1 > 0$ such that*

$$\|U(\cdot, s) g\|^2_{L^2_t(\mathbf{R}_\pm; X_E)} \leq C_1 \|g\|^2_{\mathcal{H}_E}. \tag{5.8}$$

for each $s \in \mathbf{R}_+ = (0, \infty)$ [or $\in \mathbf{R}_- = (-\infty, 0)$] and $g \in \mathcal{H}_E$.

Theorem 5.7 *Under the same conditions as above, we have*

(*i*) $\{U(t, s)\}_{t,s \in \mathbf{R}}$ *is a family of uniformly bounded operators in $\mathcal{H}_E$:*

$$\sup_{t,s \in \mathbf{R}} \|U(t, s)\|_{\mathcal{B}(\mathcal{H}_E)} = C_U < \infty.$$

(*ii*) *For every $s \in \mathbf{R}_\pm$, there exists the strong limit*

$$Z^\pm(s) = s - \lim_{t \to \pm\infty} e^{-i(-t+s)M} U(t, s).$$

(*iii*) *The operator $Z^\pm = Z^\pm(0)$ satisfies*

$$w - \lim_{s \to \pm\infty} Z^\pm U(0, s) e^{-isM} = I \quad (\textit{weak limit}).$$

(*iv*) *If ϵ_1 is chosen smaller to satisfy $\epsilon_V \sqrt{2C_0C_1} < 1$, where $\epsilon_V = \epsilon_1$ (Schrödinger), $= \max\{1, m^{-1}\}\epsilon_1$ (Klein-Gordon), then $Z^\pm : \mathcal{H} \longrightarrow \mathcal{H}_E$ is a bijection on $\mathcal{H}_E$. Thus, the scattering operator $S = Z^+(Z^-)^{-1}$ is well defined and also gives a bijection on $\mathcal{H}_E$.*

6. Proof of Theorem 5.5

Let $L = -\Delta_b$ be the selfadjoint operator in $L^2 = L^2(\Omega)$ with domain $\mathcal{D}(L) = \{u \in H^2_{\rm loc}(\Omega) \cap H^1_{b,0}(\Omega); \Delta_b u \in L^2(\Omega)\}$. For $\kappa \in \mathbf{C}_+ = \{\kappa \in \mathbf{C}; {\rm Im}\kappa > 0\}$ we put $R(\kappa^2) = (L - \kappa^2)^{-1}$. Assume $(A1)$ with sufficiently small $\epsilon_0 > 0$. Then the following two propositions hold.

Proposition 6.1 *There exist $C_2 > 0$ and $C_3 > 0$ independent of κ such that*

$$\int_\Omega (1+[r])^{-2}|R(\kappa^2)f(x)|^2 dx \le C_2 \int_\Omega (1+[r])^2 |f(x)|^2 dx,$$

$$\int_\Omega (1+[r])^{-2}\{|\nabla_b R(\kappa^2)f|^2 + |\kappa R(\kappa^2)f|^2\} dx \le C_3 \int_\Omega (1+[r])^2 |f|^2 dx.$$

For the sake of simplicity we put $\xi(r) = (1+[r]^2)^{-1/2}$.

Lemma 6.10 *Let $R_m(\kappa^2) = (L + m^2 - k^2)^{-1}$. Then there exists $C > 0$ such that*

$$(1+|\kappa|)\|\xi R_m(\kappa^2)f\| + \|\xi \nabla_b (R_m(\kappa^2)f)\| \le C\|\xi^{-1} f\|, \tag{6.1}$$

$$\|\xi \Delta_b (R_m(\kappa^2)f)\| \le C\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1} f\|\}, \tag{6.2}$$

$$|\kappa|\|\xi \nabla (R_m(\kappa^2)f)\| \le C\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1} f\|\}. \tag{6.3}$$

for each $\kappa \in \mathbf{C}_+$ and $f \in X_0'$ satisfying also $\nabla_b f \in X_0'$.

Proof Note that

$$|\kappa|^2 \|\xi R_m(k^2) f\|^2 \le m^2 \|\xi R_m(\kappa^2) f\|^2 + |-m^2 + \kappa^2| \|\xi R_m(\kappa^2) f\|^2.$$

Then (6.1) is direct from Propositions 1 and 2.

To show (6.2) we start from the equation

$$\xi \Delta_b (R_m(\kappa^2) g) = \nabla_b \cdot \{\xi \nabla_b R_m(\kappa^2) g) - (\nabla \xi) R_m(\kappa^2) g\}.$$

Put $\vec{f} = (\nabla\xi) R_m(\kappa^2) g$. Then since $\vec{f}|_{\partial\Omega} = \vec{0}$, we have

$$|(\nabla_b \cdot \vec{f}, h)| = |(\vec{f}, -\nabla_b h)| \le \|\vec{f}\| \|h\|_{\dot{H}^1_b},$$

where $\dot{H}^1_b$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla_b h\|$. Let $\dot{H}^{-1}_b$ denote the dual space of $\dot{H}^1_b$. Then we have $\|\nabla_b \cdot \vec{f}\|_{\dot{H}^{-1}_b} \le \|\vec{f}\|$, and hence noting $|\nabla \xi| \le C|\xi|$, we have

$$\|\xi \Delta_b (R_m(\kappa^2) g)\|_{\dot{H}^{-1}_b} \le \|\xi \nabla_b (R_m(\kappa^2) g)\| + \|(\nabla\xi) R_m(\kappa^2) g\| \le C\|\xi^{-1} g\|.$$

(6.2) then follows from the equality

$$(\Delta_b(R_m(\kappa^2)f,g)) = (R_m(\kappa^2)\Delta_b f,g) = (\xi^{-1}f, \xi\Delta_b(R_m(\overline{\kappa}^2)g))$$

since we have

$$\|\xi^{-1}f\|_{\dot{H}_b^1} \le \|\xi^{-1}\nabla_b f\| + \|\nabla(\xi^{-1})f\| \quad \text{and} \quad |\nabla\xi^{-1}| \le C\xi^{-1}.$$

Next note that

$$\kappa^2 R_m(\kappa^2)f = -f - (\Delta_b - m^2)R_m(\kappa^2)f.$$

Then the use of (6.1) and (6.2) shows

$$\|\xi(\Delta_b - m^2)R_m(\kappa^2)f\| \le C\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1}f\|\}.$$

Since $\|\xi f\| \le \|\xi^{-1}f\|$, this proves

$$|\kappa|^2\|\xi R_m(\kappa^2)f\| \le C\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1}f\|\}. \tag{6.4}$$

By use of (6.4), (6.1) and (6.2) we have

$$\begin{aligned}
&|\kappa|^2\|\xi\nabla_b(R_m(\kappa^2)f)\|^2 \\
&= -|\kappa|^2(\{\xi\Delta_b(R_m(\kappa^2)f) + 2\nabla\xi\cdot\nabla_b(R_m(\kappa^2)f\}, \xi R_m(\kappa^2)f) \\
&\le \{\|\xi\Delta_b(R_m(\kappa^2)f)\| + 2\|\nabla\xi\cdot\nabla_b(R_m(\kappa^2)f)\|\}|\kappa|^2\|\xi R_m(\kappa^2)f\| \\
&\quad \le C\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1}f\|\}\{\|\xi^{-1}\nabla_b f\| + \|\xi^{-1}f\|\}.
\end{aligned}$$

which proves (6.3). □

With this lemma we can prove the following proposition which attains Theorem 5.5 for M.

Proposition 6.2 *Assume* (*A*1) *with small* ϵ_0. *For* $\kappa \in \mathbf{C}\backslash\mathbf{R}$ *put* $\mathcal{R}(\kappa) = (M-\kappa)^{-1}$. *Then there exists* $C_4 > 0$ *independent of* κ *and* $f \in X_E$ *such that*

$$\|\mathcal{R}(\kappa)f\|_{X_E} \le C_4\|f\|_{X_E'}, \tag{6.5}$$

where X_E *is the weighted energy space defined by* (5.7).

Proof Note that

$$\begin{aligned}
|(\mathcal{R}(\kappa)f,g)_{\mathcal{H}_E}| = \frac{1}{2}\Big[&|(\nabla_b\{\kappa R_m(\kappa^2)f_1 + iR_m(\kappa^2)f_2\}, \nabla_b g_1) \\
&+((c+m^2)\{\kappa R_m(\kappa^2)f_1 + iR_m(\kappa^2)f_2\}, g_1)
\end{aligned}$$

$$+(\{i(\Delta_b - m^2)R_m(\kappa^2)f_1 + \kappa R_m(\kappa^2)f_2\}, g_2)|\Big] \tag{6.6}$$

$$\le \frac{1}{2}\Big[\{|\kappa|\|\xi\nabla_b(R_m(\kappa^2)f_1)\| + \|\xi\nabla_b R_m(\kappa^2)f_2\|\}\|\xi^{-1}\nabla_b g_1\|$$

$$+m^2\{|\kappa|\|\xi R_m(\kappa^2)f_1\| + \|\xi R_m(\kappa^2)f_2\|\}\|\xi^{-1}g_1\|$$

$$+\{\|\xi\Delta_b(R_m(\kappa^2)f_1)\| + m^2\|\xi R_m(\kappa^2)f_1\| + |\kappa|\|\xi R_m(\kappa^2)f_2\|\}\|\xi^{-1}g_2\|\Big].$$

Then applying the inequalities of Lemma 6.10 to each component of the right and noting $m > 0$, we see that (6.5) to hold. □

Remark The above proof is not verified so far to acoustic wave equations (i.e., in case $m = 0$). The main reason is in the difference of the energy norm. The kinetic energy which consists just of the Dirichlet norm makes difficult to apply Lemma 6.10 to acoustic wave equations.

However, as is proved in Mochizuki [//], a weighted energy estimate works well to acoustic wave equations when $n \ge 3$, and Theorem 5.6 is applied to problem (6.1) with $m^2 = 0$ if we require in place of $(BC.2)$

$$(BC.3) \qquad \left\{|\tilde{b}(x,t)\|^2 + r^2|\tilde{c}(x,t)|^2\right\}^{1/2} \le \eta(t) + \epsilon_0(1+r)^{-1-\delta}$$

for some $0 < \delta < 1$ and small $\epsilon_0 > 0$.

7. Proof of Theorem 5.6

The resolvent estimates of Theorem 5.5 lead us to the smoothing properties summarized in the following proposition.

Proposition 7.3 *Assume $(A1)$ with small ϵ_0. Then for each $h(t) \in L^2(\mathbf{R}_\pm; X')$ and $f \in \mathcal{H}$, we have*

$$\left\|\int_0^t e^{-i(t-\tau)\Lambda}h(\tau)d\tau\right\|^2_{L^2(\mathbf{R}_\pm;X)} \le C_0^2\|h\|^2_{L^2(\mathbf{R}_\pm;X')}, \tag{7.1}$$

$$\sup_{t\in\mathbf{R}_\pm}\left\|\int_0^t e^{i\tau\Lambda}h(\tau)d\tau\right\|^2_{\mathcal{H}} \le 2C_0\|h\|^2_{L^2(\mathbf{R}_\pm;X')}, \tag{7.2}$$

$$\|e^{-it\Lambda}f\|^2_{L^2(\mathbf{R}_\pm;X)} \le 2C_0\|f\|^2_{\mathcal{H}}, \tag{7.3}$$

where $\mathbf{R}_+ = (0,\infty)$ and $\mathbf{R}_- = (-\infty, 0)$.

Proof By the standard approximation procedure, we can assume $h(t) \in C_0^\infty(I; X')$ for some interval $I \subset \mathbf{R}_\pm$.

For $t \in \mathbf{R}_\pm$ we put $v(t) = \int_0^t e^{-i(t-\tau)\Lambda}h(\tau)d\tau$, where $h(t)$ is regarded to be 0 outside I, and consider its Laplace transform

$$\tilde{v}(\zeta) = \pm\int_0^{\pm\infty} e^{i\zeta t}v(t)dt, \quad \pm\text{Im}\zeta > 0,$$

Then since $\tilde{v}(\zeta) = -i\mathcal{R}(\zeta)\tilde{h}(\zeta)$, it follows from the Plancherel theorem and Theorem 1 that

$$\left|\int_I e^{\mp 2\epsilon t}(v(t), g(t))_{\mathcal{H}} dt\right| = \left|(2\pi)^{-1}\int_{-\infty}^{\infty}(\tilde{v}(\lambda \pm i\epsilon), \tilde{g}(\lambda \pm i\epsilon))_{\mathcal{H}} d\lambda\right|$$

$$\leq \int_{-\infty}^{\infty} \|\mathcal{R}(\lambda \pm i\epsilon)\tilde{h}(\lambda \pm i\epsilon)\|_X \|\tilde{g}(\lambda \pm i\epsilon)\|_{X'} d\lambda$$

$$\leq C_0 \int_I e^{\mp 2\epsilon t}\|h(t)\|_{X'}\|g(t)\|_{X'} dt$$

for any $g(t) \in C_0^\infty(I; X')$. Letting $\epsilon \downarrow 0$, we obtain inequality (7.1).

Next, note that the Fubini theorem implies

$$\left\|\int_0^t e^{is\Lambda}h(s)ds\right\|_{\mathcal{H}}^2 = \int_0^t \left(\int_0^\sigma e^{-i(\sigma-s)\Lambda}h(s)ds, h(\sigma)\right)_{\mathcal{H}} d\sigma$$

$$+ \int_0^t \left(h(s), \int_0^s e^{-i(s-\sigma)\Lambda}h(\sigma)d\sigma\right)_{\mathcal{H}} ds,$$

where $(\cdot,\cdot)_{\mathcal{H}}$ is extended to the duality between X and X'. This and (7.1) show (7.2) to hold.

(7.3) is the dual assertion of (7.2). □

Lemma 7.11 *Under* $(A2)$ *we have*

$$|(V(t)u, v)_{\mathcal{H}}| \leq \tilde{\eta}(t)\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}} + \epsilon_V\|u\|_X\|v\|_X,$$

where $\tilde{\eta}(t) = \max\{1, m^{-1}\}\eta(t)$, $\epsilon_V = \max\{1, m^{-1}\}\epsilon_1$.

Proof We have

$$|(V(t)u, v)_{\mathcal{H}_E}| = \frac{1}{2}\left|\int_\Omega \{c(x,t)u_1 + b_0(x,t)u_2\}\overline{v_2}dx\right|$$

$$\leq \frac{1}{2}\int_\Omega (\eta(t) + \epsilon_1(1+[r]^2)^{-1})\{|u_1| + |u_2|\}|v_2|dx$$

$$\leq \max\{1, m^{-1}\}\{\eta(t)\|u\|_{\mathcal{H}_E}\|v\|_{\mathcal{H}_E} + \epsilon_1\|u\|_{X_E}\|v\|_{X_E}\}.$$

Thus, the lemma also hold in this case. □

For $0 \leq \pm s \leq \pm T \leq \infty$ let $I_{+,s} = [s, T]$ or $I_{-,s} = [T, s]$. We do not exclude $T \pm \infty$ and write $\mathbf{R}_{+,s} = [s, \infty)$ or $\mathbf{R}_{-,s} = (-\infty, s]$.

With these notation let $Y(I_{\pm,s})$ be the space of functions $v(t) \in BC(I_{\pm,s}; \mathcal{H}) \cap L^2(I_{\pm,s}; X)$ (BC means the space of bouded continuous functions) such that

$$\|v\|_{Y(I_{\pm,s})} = \sup_{t \in I_{\pm,s}} \|v(t)\|_{\mathcal{H}} + \|v\|_{L^2(I_{\pm,s};X)} < \infty. \tag{34}$$

Lemma 7.12 *We put*

$$\Phi_{\pm,s}v(t) = \int_s^t e^{-i(t-s)\Lambda}V(s)v(s)ds, \quad v(t) \in Y(I_{\pm,s}).$$

Then $\Phi_{\pm,s} \in \mathcal{B}(Y(I_{\pm,s}))$ *and we have*

$$\sup_{t\in I_{\pm,s}} \|\Phi_{\pm,s}v(t)\|_{\mathcal{H}} \le \|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{t\in I_{\pm,s}} \|v(t)\|_{\mathcal{H}} + \epsilon_V\sqrt{2C_0}\|v\|_{L^2(I_{\pm,s};X)}, \tag{7.5}$$

$$\|\Phi_{\pm,s}v\|_{L^2(I_{\pm,s};X)} \le 2\sqrt{2C_0}\|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{t\in I_{\pm,s}} \|v(t)\|_{\mathcal{H}} + 3\epsilon_V C_0\|v\|_{L^2(I_{\pm,s};X)}. \tag{7.6}$$

Proof Let $g \in \mathcal{H}$. Then it follows from Lemma 7 that

$$|(\Phi_{\pm,s}v(t), g)_{\mathcal{H}}| = \left|\int_s^t (V(\tau)v(\tau), e^{-i(\tau-t)\Lambda}g)_{\mathcal{H}}d\tau\right|$$

$$\le \left|\int_s^t \tilde{\eta}(\tau)\|v(\tau)\|_{\mathcal{H}}\|g\|_{\mathcal{H}}d\tau\right| + \epsilon_V\left|\int_s^t \|v(\tau)\|_X\|e^{-i(\tau-t)\Lambda}g\|_X d\tau\right|. \tag{7.7}$$

So, by use of (7.3) and the unitarity of $e^{-it\Lambda}$ we obtain

$$|(\Phi_{\pm,s}v(t), g)_{\mathcal{H}}| \le \|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{\tau\in I_{\pm,s}} \|v(\tau)\|_{\mathcal{H}}\|g\|_{\mathcal{H}} + \epsilon_V\|v\|_{L^2(I_{\pm,s};X)}\sqrt{2C_0}\|g\|_{\mathcal{H}},$$

which implies (7.5).

Next, let $g(t) \in L^2(I_{\pm,s}; X')$. Then it similarly follows that

$$\left|\int_s^T (\Phi_{\pm,s}v(t), g(t))_{\mathcal{H}}dt\right| = \left|\int_s^T \int_s^t (V(\tau)v(\tau), e^{-i(\tau-t)\Lambda}g(t))_{\mathcal{H}}d\tau dt\right|$$

$$\le \|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{\tau\in I_{\pm,s}} \left(\|v(\tau)\|_{\mathcal{H}}\left\|\int_\tau^T e^{it\Lambda}g(t)dt\right\|_{\mathcal{H}}\right)$$

$$+\epsilon_V\|v\|_{L^2(I_{\pm,s};X)}\left\|\int_\tau^T e^{-i(\tau-t)\Lambda}g(t)dt\right\|_{L^2(I_{\pm,s};X)},$$

where

$$\left\|\int_\tau^T e^{-i(\tau-t)\Lambda}g(t)dt\right\|_{L^2(I_{\pm,s};X)} \le \left\|\int_0^\tau e^{-i(\tau-t)\Lambda}g(t)dt\right\|_{L^2(I_{\pm,s};X)}$$

$$+\left\|e^{-i\tau\Lambda}\int_0^T e^{it\Lambda}g(t)dt\right\|_{L^2(I_{\pm,s};X)}.$$

Thus, applying inequalities (7.1), (7.2) and (7.3), we obtain

$$\left|\int_s^T (\Phi_{\pm,s}v(t), g(t))_{\mathcal{H}}dt\right| \le \|\tilde{\eta}\|_{L^1(I_{\pm,s})} \sup_{\tau\in I_{\pm,s}} \|v(\tau)\|_{\mathcal{H}}2\sqrt{2C_0}\|g\|_{L^2(I_{\pm,s};X')}$$

$$+\epsilon_V \|v\|_{L^2(I_{\pm,s};X)} 3C_0 \|g\|_{L^2(I_{\pm,s};X')},$$

which implies (7.6). □

Now, since $\tilde{\eta}(t) \in L^1(\mathbf{R}_\pm)$, we can choose $0 < \delta \le 1$ and $\pm\sigma > 0$ to satisfy

$$(1 + 2\sqrt{2C_0})\|\tilde{\eta}\|_{L^1(I_{\pm,s})} < 1 \tag{7.8}$$

if $|I_{\pm,s}| = |T - s| \le \delta$ or $I_{\pm,s} = \mathbf{R}_{\pm,s}$ with $\pm s \ge \pm\sigma$. So, if ϵ_1 is chosen small enough to satisfy $\epsilon_V(2\sqrt{2C_0} + 3C_0) < 1$, then it follows from (7.4), (7.5) and (7.6) that

$$\begin{aligned}\|\Phi_{\pm,s} v\|_{Y(I_{\pm,s})} &\le \max\{(1 + 2\sqrt{2C_0})\|\tilde{\eta}\|_{L^1(I_{\pm,s})}, \epsilon_V(2\sqrt{2C_0} + 3C_0)\}\|v\|_{Y(I_{\pm,s})} \\ &< \|v\|_{Y(I_{\pm,s})}. \end{aligned} \tag{7.9}$$

Lemma 7.13 *For each fixed $I_{\pm,s}$ satisfying* (38)*, the integral equation*

$$u(t) = e^{-i(t-s)\Lambda} f - i \int_s^t e^{-i(t-\tau)\Lambda} V(\tau) u(\tau) d\tau \tag{7.10}$$

has a solution $u(t) \in Y(I_{\pm,s})$ and it satisfies

$$\|u\|_{Y(I_{\pm,s})} = \sup_{t \in I_{\pm,s}} \|u(t)\|_{\mathcal{H}} + \|u\|_{L^2(I_{\pm,s};X)} \le C_{\delta,\sigma} \|f\|_{\mathcal{H}} \tag{7.11}$$

for some $C_{\delta,\sigma} > 0$ independent f.

Proof We define $\{u_k(t)\}$ successively as follows:

$$u_0(t) = e^{-i(t-s)\Lambda} f, \quad u_k(t) = u_0(t) - i\Phi_{\pm,s} u_{k-1}(t).$$

Note that the unitarity of $e^{-it\Lambda}$ and (7.3) show

$$\|u_0\|_{Y(I_{\pm,s})} = \|u_0(t)\|_{\mathcal{H}} + \|u_0\|_{L^2(I_{\pm,s};X)} \le (1 + \sqrt{2C_0})\|f\|_{\mathcal{H}}. \tag{7.12}$$

Thus, $u_0(t) \in Y(I_s)$ and also each $u_k(t) \in Y(I_{\pm,s})$. Since

$$\|u_k - u_{k-1}\|_{Y(I_{\pm,s})} \le \left(\|\Phi_{\pm,s}\|_{\mathcal{B}(Y_{I_{\pm,s}})}\right)^k \|u_0\|_{Y_{I_{\pm,s}}}, \tag{7.13}$$

we see from (39) that

$$u_n(t) = u_0(t) + \sum_{k=1}^n \{u_k(t) - u_{k-1}(t)\}$$

converges in $Y_{I_{\pm,s}}$ as $n \to \infty$,. The limit $u(t)$ obviously solves the integral equation (7.10). Inequality (7.11) with

$$C_{\delta,\sigma} = \frac{1 + \sqrt{2C_0}}{1 - \|\Phi_{\pm,s}\|_{\mathcal{B}(Y(I_{\pm,s}))}}$$

is a result of (7.12) and (7.13). □

Proof of Theorem 5.7 For δ and $\pm\sigma$ given in (7.8) we choose integer N to satisfy $N\delta \geq \pm\sigma$, and divide $\mathbf{R}_\pm$ into $N+1$ subintervals

$$I_{+,s_j} = [s_j, s_{j+1}] \text{ or } I_{-,s_j} = [s_{j+1}, s_j] \ (j = 0, 1, \cdots, N-1), \text{ and } I_{\pm,s_N} = \mathbf{R}_{\pm,s_N},$$

where $s_j = \pm j\delta$ $(j = 0, 1, \cdots, N)$. Then by Lemma 9 the solution of (7.10) with $f = u(s_j)$ is constructed in each interval $I_{\pm,s_j}$, and by putting together, a global solution of (5.6) is obtained. Moreover, the above argument and (7.11) imply (5.8) to hold with $C_1 = (N+1)C_{\delta,\sigma}^N$.

The uniqueness of solutions in $C(\mathbf{R};\mathcal{H})$ follows from the inequality

$$\|\Phi_{\pm,s}v(t)\|_{\mathcal{H}} \leq \left|\int_s^t \|V(\tau)v(\tau)\|_{\mathcal{H}} d\tau\right| \leq \left|\int_s^t \{\tilde{\eta}(\tau) + \epsilon_1\} v(\tau)\|v(\tau)\| d\tau\right|$$

$$\leq \{\|\tilde{\eta}\|_{L^1(I_{\pm,s,t})} + \epsilon_1|I_{\pm,s,t}|\} \sup_{t\in I_{\pm,s,t}} \|v(t)\|_{\mathcal{H}},$$

where $I_{+,s,t} = (s,t)$ when $0 \leq s < t$ and $I_{-,s,t} = (t,s)$ when $t < s < 0$, since we can choose $\delta = |t-s|$ small enough to satisfy

$$\|\tilde{\eta}\|_{L^1(I_{\pm,s,t})} + \epsilon_1|I_{\pm,s,t}| < 1.$$

□

5. Proof of Theorem 5.7

Proof of Theorem 5.7 will be based on Lemma 7 and inequalities of Proposition 4 and Theorem 5.6.

We put $u(t,s) = U(t,s)f$, $u_0(t-s) = e^{-i(t-s)\Lambda}f_0$. Then we have from (5.6)

$$(u(t,s), u_0(t-s))_{\mathcal{H}} = (f, f_0)_{\mathcal{H}} - i\int_s^t (V(\tau)u(\tau,s), u_0(\tau-s))_{\mathcal{H}} d\tau.$$

In the right side we apply the inequality of Lemma 7. It then follows from (7.3) and (5.8) that for any σ, $t \in \mathbf{R}_\pm$,

$$|(u(t,s), u_0(t-s))_{\mathcal{H}} - (u(\sigma,s), u_0(\sigma-s))_{\mathcal{H}}| \leq \left|\int_\sigma^t \tilde{\eta}(\tau)\|u(\tau,s)\|_{\mathcal{H}}\|u_0(\tau-s)\|_{\mathcal{H}} d\tau\right|$$

$$+\epsilon_V \left|\int_\sigma^t \|u(\tau,s)\|_X^2 d\tau\right|^{1/2} \left|\int_\sigma^t \|u_0(\tau-s))\|_X^2 d\tau\right|^{1/2}. \tag{8.1}$$

All the assertions of the theorem are verified from this inequality.

Proof of Theorem 3 (*i*) We put $\sigma = s$ in (7.14). Then by (7.3) and (5.8)

$$|(u(t,s), u_0(t-s))_{\mathcal{H}} - (f, f_0)_{\mathcal{H}}| \leq \left|\int_s^t \tilde{\eta}(\tau)\|u(\tau,s)\|_{\mathcal{H}}\|u_0(\tau-s)\|_{\mathcal{H}} d\tau\right|$$

$$+\epsilon_V\sqrt{2C_0C_1}\|f\|_{\mathcal{H}}\|f_0\|_{\mathcal{H}}.$$

Since $e^{-i(t-s)\Lambda}$ is unitary, it follows that

$$\|u(t,s)\|_{\mathcal{H}} \le (1+\epsilon_V\sqrt{2C_0C_1})\|f\|_{\mathcal{H}} + \int_s^t \tilde{\eta}(\tau)\|u(\tau,s)\|_{\mathcal{H}}d\tau.$$

The requirement $\eta(t)\in L^1(\mathbf{R})$ and the Gronwall inequality show the assertion with

$$C_U = (1+\epsilon_V\sqrt{2C_0C_1})e^{\|\tilde{\eta}\|_{L^1}}.$$

(ii) Noting (i), we have from (7.14), (7.3) and (5.8)

$$|(u(t,s),u_0(t-s))_{\mathcal{H}} - (u(\sigma,s),u_0(\sigma-s))_{\mathcal{H}}| \le \Big\{C_U\|f\|_{\mathcal{H}}\Big|\int_\sigma^t \tilde{\eta}(\tau)d\tau\Big| +$$

$$+\epsilon_V\Big|\int_\sigma^t \|u(\tau,s)\|_X^2 d\tau\Big|^{1/2}\sqrt{2C_0}\Big\}\|f_0\|_{\mathcal{H}}.$$

Here, for fixed any $s\in\mathbf{R}_\pm$, $\Big\{\cdots\Big\}\to 0$ as $\sigma,\ t\to\pm\infty$. Thus, $e^{-i(s-t)\Lambda}U(t,s)$ converges strongly in $\mathcal{H}$ as $t\to\pm\infty$.

(iii) Let $\sigma=s$ and $t\to\pm\infty$ in (7.14). Then noting (i) and (5.8), we have

$$|(Z^\pm(s)f,f_0)_{\mathcal{H}} - (f,f_0)_{\mathcal{H}}| \le \|f\|_{\mathcal{H}}\Big\{C_U\Big|\int_s^{\pm\infty}\tilde{\eta}(\tau)d\tau\Big|\|f_0\|_{\mathcal{H}} +$$

$$+\epsilon_V\sqrt{C_1}\Big|\int_s^{\pm\infty}\|u_0(\tau-s))\|_X^2 d\tau\Big|^{1/2}\Big\}. \tag{8.2}$$

Choose here $f=e^{-is\Lambda}g$ and $f_0=e^{-is\Lambda}g_0$. Then

$$|(\{e^{is\Lambda}Z^\pm(s)e^{-is\Lambda}-I\}g,g_0)_{\mathcal{H}}| \le \|e^{-is\Lambda}g\|_{\mathcal{H}}\Big\{C_U\Big|\int_s^{\pm\infty}\tilde{\eta}(\tau)d\tau\Big|\|e^{-is\Lambda}g_0\|_{\mathcal{H}} +$$

$$+\epsilon_V\sqrt{2C_0}\Big|\int_s^{\infty}\|e^{-i\tau\Lambda}g_0\|_X d\tau\Big|^{1/2}\Big\}.$$

g and g_0 being arbitrary, this implies that as $s\to\pm\infty$,

$$Z^\pm U(0,s)e^{-is\Lambda} = e^{is\Lambda}Z^\pm(s)e^{-is\Lambda}\to I \ \text{ weakly in } \mathcal{H}.$$

(iv) Note that (7.15) and (7.3) imply

$$|(\{Z^\pm(s)-I\}f,f_0)_{\mathcal{H}}| \le \Big\{\Big|\int_s^{\pm\infty}\tilde{\eta}(\tau)d\tau\Big|C_U + \epsilon_V\sqrt{2C_0C_1}\Big\}\|f\|_{\mathcal{H}}\|f_0\|_{\mathcal{H}}.$$

Since $\epsilon_V\sqrt{2C_1C_0}<1$, we can choose $\pm s>0$ sufficiently large to satisfy

$$\Big|\int_s^{\pm\infty}\tilde{\eta}(\tau)d\tau\Big|C_U + \epsilon_V\sqrt{2C_0C_1} < 1.$$

Thus, $\|Z_{\pm}(s)-I\|_{\mathcal{B}(\mathcal{H})}<1$ and $Z^{\pm}(s)$ gives a bijection on $\mathcal{H}$. The same property of $Z^{\pm}$ then easily follows. □

Final Remarks

In case $\Omega=\mathbf{R}^n$ $(n\geq 3)$ and $b(x)=0$, similar results have been obtained in [11] and [13], for complex potentials satisfying

$$c(x,t)\in L^{\nu}(\mathbf{R};L^p)\cap BC(\mathbf{R}^{n+1})$$

with

$$0\leq \frac{1}{p}\leq \frac{2}{n} \quad \text{and} \quad \frac{1}{\nu}=1-\frac{n}{2p}.$$

The smallness condition

$$\|c\|_{L^{\infty}(\mathbf{R}_{\pm};L^{n/2})}<<1$$

is also required when $\nu=\infty$.

The arguments employed in these works are based on the Fourier transformation, and are not directly applicable to the problems in exterior domain. Moreover, note that the function

$$c(x,t)=c_0(1+r)^{-\alpha}(1+|t|)^{-\beta} \tag{46}$$

with α, $\beta\geq 0$ satisfies $(A2)$ and also the above conditions if $\alpha/2+\beta>1$. However, the function

$$c(x,t)=c_0\sin t(1+r)^{-2} \quad \text{with small } |c_0|>0$$

satisfies $(A2)$ but slips out of the above conditions.

The potential (46) has been considered in Yafaev [17] when c is real and $\beta>0$. For the Schrödinger equation (1) in $\mathbf{R}^n$ $(n\geq 3)$ with $b=0$ his results include the following. The wave operator

$$W^{\pm}=s-\lim_{t\to\pm\infty}U(0,t)e^{itL}$$

exists if $\alpha+\beta>1$. It is in general incomplete, but becomes complete, i.e., the range of $W^{\pm}$ coincides with the whole space $L^2(\mathbf{R}^n)$, if the stronger condition $\dfrac{\alpha}{2}+\beta>1$ is required.

References

[1] T. Hoshiro, *Decay and regularity of dispersive equations with constant coefficients*, J. Anal. Math., **91** (2003), 211-230.

[2] R. Ikehata, G. Todorova and B. Yordanov, *Optimal decay rate og the energy for wave equations with critical potenitial*, J. Math. Soc. Japan **65** (2013), 183-136.

[3] T. Kato and K. Yajima, *Some examples of smooth operators and associated smoothing effect*, Rev. Marth. Phys., **1** (1989), 481-496.

[4] A. Matsumura, *Energy decay of dissipative wave equations*, Proc. Japan Acad., Ser. A **53** (1977), 232-236.

[5] K. Mochizuki, *Scattering theory for the wave equations with dissipative terms*, Publ. Res. Inst. Math. Sci., **12** (1976), 283-290.

[6] K. Mochizuki, *On scattering for wave equations with time dependent coefficients*, Tsukuba J. Math., **31** (2007), 327-341.

[7] K. Mochizuki, *Uniform Resolvent estimates for magnetic Schrödinger operators and smoothing effects for related evolution equations*, Publ. Res. Inst. math. Sci., **46** (2010), 741-754.

[8] K. Mochizuki and T. Motai, *On energy decay-nondecay problems for wave equations with nonlinear dissipative term in* $\mathbf{R}^n$, J. Math. Soc. Japan, **47** (1995), 405-421.

[9] K. Mochizuki and S. Murai, *Smoothing properties and scattering for Klein-Gordon equations in exterior domain with time dependent perturbations*, Publ. Res. Inst. Math. Sci., **53** to appear.

[10] K. Mochizuki and H. Nakazawa, *Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation*, Publ. Res. Inst. Math. Sci., **32** (1996), 401-414.

[11] K. Mochizuki and H. Nakazawa, *Uniform resolvent estimates for magnetic Schrödinger operators in 2D exterior domain and their applications to related evolution equatios*, Publ. Res. Inst. Math. Sci., **51** (2015), 319-336.

[12] T. Motai, *Decay and asymptotics for wave equations with nonlinear dissipative terms*, Thesis (University of Tsukuba), 1993.

[13] T. Motai and K. Mochizuki, *Asymptotic behavior of solutions to the wave equation with nonlinear dissipative term in* $\mathbf{R}^n$, Advances Math. Sci. Appl., **19** (2009), 211-225.

[14] K. Nishihara, *Decay Properties for the damped wave equation with space dependent potential and absorbed semilinear term*, Comm. Partial Differential Equations, **35** (2010), 1402-1418.

[15] R. S. Strichartz, *Convolution with kernels having singularities on a sphere*, Trans. Amer. Math. Soc., **148** (1970), 460-471.

[16] M. Sugimoto, *Global smoothing properties of generalized Schrödinger equations*, J. Anal. Math., **76** (1998), 191-204.

[17] Y. Wakasugi, *On diffusion phenomena for the linear wave quation with space dependent damping*, J. Hyp. Diff. Eqs. **11** (2014), 795-818.

[18] D. R. Yafaev, Mathematical Scattering Theory, Analytic Theory, AMS, Providence, Rhode Island, Mathematical Surveys and Monographs **158**, 2009.