Direct and inverse bifurcation problems and related topics I

Tetsutaro Shibata

Laboratory of Mathematics, Graduate School of Engineering

Hiroshima University

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Outline

Introduction

We first consider

$$
\begin{aligned}
-u''(t) &= \lambda\left(u(t) + g(u(t))\right), \quad t \in I := (-1,1), && (1.1)\\
u(t) &> 0, \quad t \in I, && (1.2)\\
u(-1) &= u(1) = 0, && (1.3)
\end{aligned}
$$

where $\lambda > 0$ is a parameter, and in what follows, we assume that $g(u)$ satisfies the following conditions.

(A.1) $g(u) \in C^1(\mathbb{R})$ and $u + g(u) > 0$ for $u > 0$.

(A.2) $g(u + 2\pi) = g(u)$ for $u \in \mathbb{R}$.

How to construct a solution and a bifurcation curve

It is well known (cf. [T. Laetsch, 1970]) that if

$$u + g(u) > 0 \qquad \text{for} \quad u > 0,$$

then by **time-map method**, we find that λ is parameterized by using $\alpha = \|u\|_\infty$, such as $\lambda = \lambda(\alpha)$ and is a continuous function of $\alpha > 0$. Since λ depends on g, we sometimes write

$$\lambda = \lambda(g, \alpha).$$

How to construct a solution and a bifurcation curve

We put

$$f(u) := u + g(u), \quad F(u) := \int_0^u f(s)ds.$$

If (u, λ) is a solution of (1.1)–(1.3) with $\|u\|_\infty = \alpha > 0$, then by (1.1), we have

$$\{u''(t) + \lambda f(u(t))\}u'(t) = 0.$$

So for $-1 \le t \le 0$,

$$\frac{1}{2}u'(t)^2 + \lambda F(u(t)) = \text{constant} = \lambda F(\alpha).$$

Since $u'(t) \ge 0$ for $-1 \le t \le 0$, we have

$$u'(t) = \sqrt{2\lambda}\sqrt{F(\alpha) - F(u(t))} \quad (-1 \le t \le 0).$$

Then by putting $u(t) = \theta$, we obtain

$$\sqrt{2\lambda} = \int_{-1}^{0} \frac{u'(t)}{\sqrt{F(\alpha) - F(u(t))}} dt = \int_{0}^{\alpha} \frac{1}{\sqrt{F(\alpha) - F(\theta)}} d\theta. \quad (1.4)$$

Therefore, for any given constant $\alpha > 0$, we define $\lambda(\alpha)$ by (1.4). Then the equation

$$(t+1)\sqrt{2\lambda(\alpha)} = \int_{0}^{u} \frac{1}{\sqrt{F(\alpha) - F(\theta)}} d\theta$$

defines a one-to-one relation between t and u for $-1 \le t \le 0$ and $0 \le u \le \alpha$ so that $t = -1$ if $u = 0$ and $t = 0$ if $u = \alpha$. Then by the function $u(t)$ so defined and using a reflection with respect to $t = 0$, we can construct a C^2-solution of (1.1)–(1.3).

Purpose

・The study of the structures of the bifurcation curves is one of the main topics in bifurcation analysis, and there are quite many works concerning the properties of bifurcation diagrams.

・In particular, the qualitative properties of the oscillatory bifurcation diagrams have been studied intensively.

・In this talk, we focus on the study whether $\lambda(g, \alpha)$ inherits the oscillatory properties of $g(u)$ or not if $g(u)$ is a periodic function.

Example

To clarify our intention, we first consider the typical example

$$g_0(u) = (1/2)\sin u,$$

which satisfies (A.1)–(A.2). Recently, the following asymptotic formula for $\lambda(g_0, \alpha)$ as $\alpha \to \infty$ has been obtained in [S, 2016].

Theorem 1.0 ([S, 2016]). *Let* $g_0 = (1/2)\sin u$. *Then as* $\alpha \to \infty$,

$$\lambda(g_0, \alpha) = \frac{\pi^2}{4} - \frac{\pi}{2\alpha}\sqrt{\frac{\pi}{2\alpha}}\sin\left(\alpha - \frac{1}{4}\pi\right) + O(\alpha^{-2}). \quad (1.5)$$

We see from Theorem 1.0 that $\lambda(g_0, \alpha)$ satisfies the following oscillatory property (OP).

(OP) $\lambda(g,\alpha) \to \pi^2/4$ *as* $\alpha \to \infty$, *and it intersects the line* $\lambda = \pi^2/4$ *infinitely many times for* $\alpha \gg 1$.

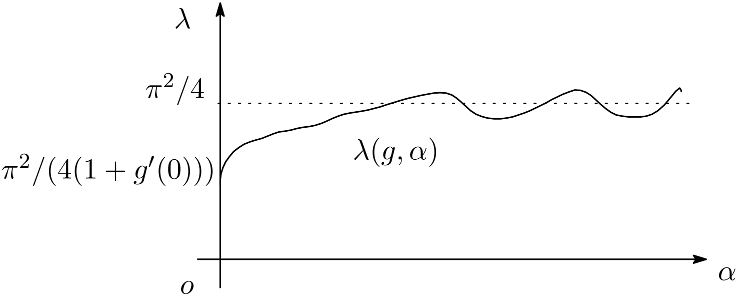


Fig. 1: $\lambda(g,\alpha)$ **with (OP)** $(g(0)=0)$

Example

Since $g(u)$ is bounded in $\mathbb{R}$ by (A.2), it is clear that $\lambda(g,\alpha) \to \pi^2/4$ as $\alpha \to \infty$. Therefore, the essential point is to find the condition whether $\lambda(g,\alpha)$ intersects the line $\lambda = \pi^2/4$ infinitely many times for $\alpha \gg 1$. For example, we have

- $g(u) = \dfrac{1}{2}\sin^{2n+1}(u) \Longrightarrow$ (OP)
- $g(u) = \dfrac{1}{2}\sin^{2n}(u) \Longrightarrow$ Not (OP)

Indeed,

First Results

Theorem 1.1 ([S, 2016]). (i) *Let* $k = 2n$ $(n \geq 1)$. *Then as* $\alpha \to \infty$

$$\lambda(2n, \alpha) = \frac{\pi^2}{4} - \frac{\pi}{2^{2n+1}\alpha}\binom{2n}{n} - \frac{\pi^{3/2}}{2^{2n+1}\alpha^{3/2}}\sum_{r=0}^{n-1}(-1)^{n-r}\binom{2n}{r} \times \frac{1}{\sqrt{n-r}}\sin\left((2n-2r)\alpha + \frac{\pi}{4}\right) + O(\alpha^{-2}). \quad (1.6)$$

(ii) *Let* $k = 2n + 1$ $(n \geq 0)$. *Then as* $\alpha \to \infty$

$$\lambda(2n+1, \alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{2^{2n+1}\alpha^{3/2}}\sum_{r=0}^{n}(-1)^{n+r}\binom{2n+1}{r} \times \sqrt{\frac{1}{2(2n-2r+1)}}\sin\left((2n-2r+1)\alpha - \frac{1}{4}\pi\right) + O(\alpha^{-2}). \quad (1.7)$$

when $k = 2n$

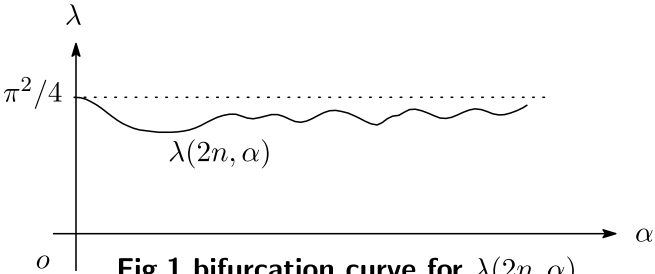


Fig.1 bifurcation curve for $\lambda(2n, \alpha)$

when $k = 2n + 1$

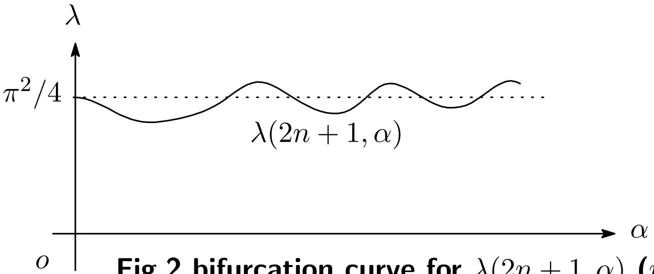


Fig.2 bifurcation curve for $\lambda(2n+1,\alpha)$ ($n \geq 1$)

Delicate Example

・For instance, we consider the following typical example. Let $\delta, \epsilon > 0$ be small fixed constants. We consider $\psi \in C^1(\mathbb{R})$ satisfying

$$\begin{aligned} \psi(t) &> 0, \qquad t \in I_\delta := (\pi/2 - \delta, \pi/2 + \delta), \\ \psi(t) &= 0, \qquad [-\pi, \pi] \setminus I_\delta, \end{aligned}$$

and

$$\begin{aligned} g_\epsilon(u) &:= \sin u + \epsilon\psi(u) \text{ for } u \in [-\pi, \pi], \\ g_\epsilon(u + 2\pi) &= g_\epsilon(u) \text{ for } u \in \mathbb{R}. \end{aligned}$$

Clearly, $g_\epsilon(u)$ satisfies (A.1)–(A.2). However, it seems difficult to distinguish whether $g(u)$ satisfies (OP) or not.

Graph of g_ϵ

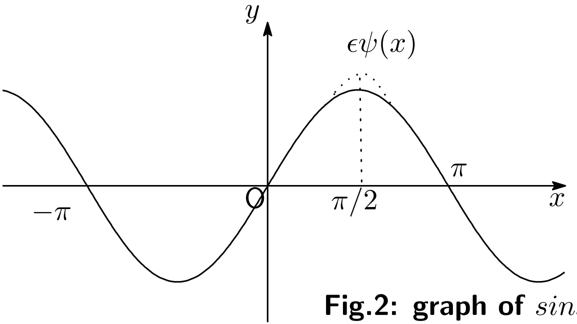


Fig.2: graph of $sinx + \epsilon\psi(x)$

Now we state our main results.

Global behavior of bifurcation curve

Theorem 1.2. *Assume that $g(u)$ satisfies (A.1)–(A.2). Then as $\alpha \to \infty$,*

$$\lambda(g,\alpha) = \frac{\pi^2}{4} - \frac{\pi a_0}{2\alpha} - \frac{1}{\alpha}\sqrt{\frac{\pi}{2\alpha}}\sum_{n=1}^{\infty}\frac{c_n}{n^{3/2}} + O(\alpha^{-2}), \tag{1.8}$$

where

$$a_0 := \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)d\theta, \tag{1.9}$$

$$c_n := \int_{-\pi}^{\pi} g'(\theta)\cos\left(n(\theta-\alpha)+\frac{3}{4}\pi\right)d\theta, \quad (n \in \mathbb{N}). \tag{1.10}$$

Corollary

As a corollary of Theorem 1.2, we obtain a meaningful result for the asymptotic property of $\lambda(g, \alpha)$.

Corollary 1.3. *Assume that $g(u)$ satisfies (A.1)–(A.2). If $a_0 \neq 0$, then $\lambda(g, \alpha)$ does not satisfy (OP).*

We apply Corollary 1.3 to $\lambda(g_\epsilon, \alpha)$. In this case, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g_\epsilon(\theta)d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi}(\sin\theta + \epsilon\psi(\theta))d\theta = \frac{\epsilon *}{\pi}\int_{-\pi}^{\pi}\psi(\theta)d\theta > 0.$$

By this, $\lambda(g_\epsilon, \alpha)$ does not satisfy (OP).

Local behavior of bifurcation curve

Remark. Theorem 1.2 is also useful to determine $g(u)$ satisfies (OP). For instance, let $g_1(u) = (\sin u + \sin 2u)/4$. We show that $\lambda(g_1, u)$ satisfies (OP) in Example 2.2 in Section 2.

Local behavior of $\lambda(g, \alpha)$.

The method to study the local behavior of $\lambda(\alpha)$ has been already established in [S, 2014,2016], since the time map method and Taylor expansion work very well in this case. To understand the total structure of $\lambda(g, \alpha)$, we show the following asymptotic formulas for completeness.

Local behavior of bifurcation curve

Theorem 1.4. *Assume (A.1)–(A.2). Furthermore, assume that $g \in C^2$ near $u = 0$.*

(i) Assume that $g(0) \neq 0$. Then as $\alpha \to 0$,

$$\lambda(g, \alpha) = \frac{2\alpha}{g(0)} \left\{1 + A_1\alpha + A_2\alpha^2 + o(\alpha^2)\right\}, \qquad (1.11)$$

where

$$A_1 = -\frac{5}{6g(0)}(1 + g'(0)), \quad A_2 = \frac{32}{45}\frac{(1 + g'(0))^2}{g(0)^2} - \frac{11}{30}\frac{g''(0)}{g(0)}. \qquad (1.12)$$

(ii) Assume that $g(0) = 0$ and $g'(0) > -1$. Then as $\alpha \to 0$,

$$\lambda(g, \alpha) = \frac{1}{1 + g'(0)}\left(\frac{\pi^2}{4} - \frac{\pi g''(0)}{3(1 + g'(0))}\alpha + o(\alpha)\right). \qquad (1.13)$$

Rough graph of $\lambda(\alpha)$ with (OP)

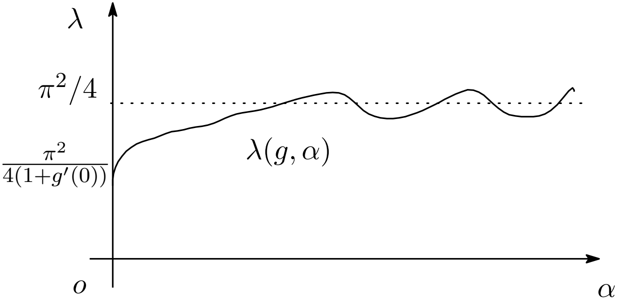


Fig. 3: $\lambda(g,\alpha)$ **with (OP)**

Proof of Theorem 1.2: Global behavior of $\lambda(\alpha)$

・The proof of Theorem 1.2 is given by the combination of **time-map method, Fourier expansion and the asymptotic formulas for some special functions**.

・In this section, let $\alpha \gg 1$. For simplicity, we write $\lambda = \lambda(g, \alpha)$. Furthermore, we denote by C the various positive constants independent of α. We put

$$G(u) := \int_0^u g(s)ds. \tag{2.1}$$

It is known that if $(u_\alpha, \lambda) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1.1)–(1.3), then

$$u_\alpha(t) = u_\alpha(-t), \quad 0 \le t \le 1, \tag{2.2}$$

$$u_\alpha(0) = \max_{-1 \le t \le 1} u_\alpha(t) = \alpha, \tag{2.3}$$

$$u_\alpha'(t) > 0, \quad -1 < t < 0. \tag{2.4}$$

time-map

We construct the well known time-map. By (1.1), we have

$$\left\{u_\alpha''(t) + \lambda\left(u_\alpha(t) + g(u_\alpha(t))\right)\right\} u_\alpha'(t) = 0.$$

By this and putting $t = 0$, we obtain

$$\frac{1}{2}u_\alpha'(t)^2 + \lambda\left(\frac{1}{2}u_\alpha(t)^2 + G(u_\alpha(t))\right) = \text{constant} = \lambda\left(\frac{1}{2}\alpha^2 + G(\alpha)\right).$$

This along with (2.4) implies that for $-1 \le t \le 0$,

$$u_\alpha'(t) = \sqrt{\lambda}\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}. \tag{2.5}$$

It follows from (A.2) that $|g(u)| \le C$ for $u \in \mathbb{R}$. Then for $0 \le s \le 1$,

$$\left|\frac{G(\alpha) - G(\alpha s)}{\alpha^2(1 - s^2)}\right| = \left|\frac{\int_{\alpha s}^{\alpha} g(t)dt}{\alpha^2(1 - s^2)}\right| \le \frac{C\alpha(1 - s)}{\alpha^2(1 - s^2)} \le C\alpha^{-1}. \tag{2.6}$$

time map

By (2.5), (2.6), putting $s := u_\alpha(t)/\alpha$ and Taylor expansion, we obtain

$$\begin{aligned}
\sqrt{\lambda} &= \int_{-1}^{0} \frac{u_\alpha'(t)}{\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}} dt \qquad (2.7)\\
&= \int_0^1 \frac{1}{\sqrt{1 - s^2 + 2(G(\alpha) - G(\alpha s))/\alpha^2}} ds \\
&= \int_0^1 \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1 + 2(G(\alpha) - G(\alpha s))/(\alpha^2(1-s^2))}} ds \\
&= \int_0^1 \frac{1}{\sqrt{1-s^2}} \left\{ 1 - \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)} + O(\alpha^{-2}) \right\} ds \\
&:= \frac{\pi}{2} - \frac{1}{\alpha^2} K(\alpha) + O(\alpha^{-2}),
\end{aligned}$$

where

$$K(\alpha) := \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1-s^2)^{3/2}} ds. \qquad (2.8)$$

We calculate $K(\alpha)$ by using the asymptotic formulas for some special functions. It is known that under the conditions (A.1)–(A.2),

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{2.9}$$

holds for $x \in \mathbb{R}$ and the right hand side of (2.9) converges to $g(x)$ uniformly on $\mathbb{R}$. Here,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos n\theta d\theta = -\frac{1}{n\pi}\int_{-\pi}^{\pi} g'(\theta)\sin n\theta d\theta \quad (2.10) \\ &:= -\frac{1}{n}\tilde{a}_n \quad (n \in \mathbb{N}_0), \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin n\theta d\theta = \frac{1}{n\pi}\int_{-\pi}^{\pi} g'(\theta)\cos n\theta d\theta \quad (2.11) \\ &:= \frac{1}{n}\tilde{b}_n \quad (n \in \mathbb{N}). \end{aligned}$$

We obtain (2.10) and (2.11) by using integration by parts, since $g(-\pi) = g(\pi)$ by (A.2).

Lemma 2.1. *As* $\alpha \to \infty$,

$$K(\alpha) \;=\; \frac{1}{2}a_0\alpha + \frac{1}{\pi}\sqrt{\frac{\pi\alpha}{2}}\sum_{n=1}^{\infty}\frac{c_n}{n^{3/2}} + O(\alpha^{-1/2}). \qquad (2.12)$$

Proof. We put $s = \sin\theta$ in (2.8). Then by integration by parts, we obtain

$$\begin{aligned} K(\alpha) &= \int_0^{\pi/2}\frac{1}{\cos^2\theta}(G(\alpha) - G(\alpha\sin\theta))d\theta \qquad (2.13) \\ &= \int_0^{\pi/2}(\tan\theta)'(G(\alpha) - G(\alpha\sin\theta))d\theta \\ &= [\tan\theta(\, G(\alpha) - G(\alpha\sin\theta))]_0^{\pi/2} + \alpha\int_0^{\pi/2} g(\alpha\sin\theta)\sin\theta d\theta. \end{aligned}$$

Asymptotic behavior of $K(\alpha)$

By l'Hôpital's rule, we obtain

$$\lim_{\theta\to\pi/2} \frac{G(\alpha) - G(\alpha\sin\theta)}{\cos\theta} = \lim_{\theta\to\pi/2} \frac{\alpha g(\alpha\sin\theta)\cos\theta}{\sin\theta} = 0. \quad (2.14)$$

For $n = \mathbb{N}$, we put

$$U_n := \int_0^{\pi/2} \cos(n\alpha\sin\theta)\sin\theta d\theta, \quad (2.15)$$

$$V_n := \int_0^{\pi/2} \sin(n\alpha\sin\theta)\sin\theta d\theta. \quad (2.16)$$

By (2.13)–(2.16), we obtain

$$
\begin{aligned}
K(\alpha) &= \alpha \int_0^{\pi/2} g(\alpha \sin\theta) \sin\theta d\theta \qquad (2.17) \\
&= \alpha \int_0^{\pi/2} \left\{ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(n\alpha \sin\theta) \right. \\
&\qquad \left. + \sum_{n=1}^{\infty} b_n \sin(n\alpha \sin\theta) \right\} \sin\theta d\theta \\
&= \alpha \left\{ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \int_0^{\pi/2} \cos(n\alpha \sin\theta) \sin\theta d\theta \right. \\
&\qquad \left. + \sum_{n=1}^{\infty} b_n \int_0^{\pi/2} \sin(n\alpha \sin\theta) \sin\theta d\theta \right\} \\
&= \alpha \left\{ \frac{1}{2} a_0 - \sum_{n=1}^{\infty} \frac{1}{n} \tilde{a}_n U_n + \sum_{n=1}^{\infty} \frac{1}{n} \tilde{b}_n V_n \right\}.
\end{aligned}
$$

Asymptotic behavior of $K(\alpha)$

Put $\theta = \pi/2 - \phi$ in (2.15). Then by (2.9)–(2.12), (2.14), (2.15) and [I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products, p.425], we obtain ($\mathbf{E}_\nu(z)$: Weber functions, $Y_\nu(z)$: Neumann functions)

$$\begin{aligned} U_n &= \int_0^{\pi/2} \cos(n\alpha\cos\phi)\cos\phi\, d\phi \qquad (2.18)\\ &= \frac{\pi}{4}(\mathbf{E}_1(n\alpha) - \mathbf{E}_{-1}(n\alpha))\\ &= \frac{\pi}{4}(-Y_1(n\alpha) + Y_{-1}(n\alpha) + O((n\alpha)^{-2})\\ &= \frac{\pi}{4}\left(-\sqrt{\frac{2}{n\pi\alpha}}\sin\left(n\alpha - \frac{3}{4}\pi\right) + \sqrt{\frac{2}{n\pi\alpha}}\sin\left(n\alpha + \frac{1}{4}\pi\right)\right)\\ &\quad + O((n\alpha)^{-3/2})\\ &= -\sqrt{\frac{\pi}{2n\alpha}}\sin\left(n\alpha - \frac{3}{4}\pi\right) + O((n\alpha)^{-3/2}), \end{aligned}$$

Asymptotic behavior of $K(\alpha)$

$$
\begin{aligned}
V_n &= \int_0^{\pi/2} \sin(n\alpha\cos\phi)\cos\phi\, d\phi \qquad (2.19)\\
&= \frac{\pi}{4}\{\mathbf{J}_1(n\alpha) - \mathbf{J}_{-1}(n\alpha)\}\\
&= \frac{\pi}{4}\{J_1(n\alpha) - J_{-1}(n\alpha)\}\\
&= \frac{\pi}{4}\left\{\sqrt{\frac{2}{n\pi\alpha}}\cos\left(n\alpha - \frac{3}{4}\pi\right) - \sqrt{\frac{2}{n\pi\alpha}}\cos\left(n\alpha + \frac{1}{4}\pi\right)\right\}\\
&\quad + O((n\alpha)^{-3/2})\\
&= \sqrt{\frac{\pi}{2n\alpha}}\cos\left(n\alpha - \frac{3}{4}\pi\right) + O((n\alpha)^{-3/2}).
\end{aligned}
$$

($\mathbf{J}_\nu(z)$: Anger functions, $J_\nu(z)$: Bessel functions)

Asymptotic behavior of $K(\alpha)$

By (2.15)–(2.19), we obtain

$$\begin{aligned} K(\alpha) &= \alpha\left\{\frac{1}{2}a_0 + \sqrt{\frac{\pi}{2\alpha}}\sum_{n=1}^{\infty}\left(\tilde{a}_n \sin\left(n\alpha - \frac{3}{4}\pi\right)\right.\right. \\ &\qquad \left.\left. + \tilde{b}_n \cos\left(n\alpha - \frac{3}{4}\pi\right)\right)\frac{1}{n^{3/2}}\right\} \\ &\quad + O\left(\alpha^{-1/2}\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}\right) \\ &= \alpha\left\{\frac{1}{2}a_0 + \frac{1}{\pi}\sqrt{\frac{\pi}{2\alpha}}\sum_{n=1}^{\infty}\frac{c_n}{n^{3/2}}\right\} + O(\alpha^{-1/2}). \end{aligned}$$

Thus the proof is complete. □

By (2.7) and Lemma 2.1, we obtain Theorem 1.2. □

Special functions and their asymptotic behavior

$J_\nu(z)$: Bessel functions, $Y_\nu(z)$: Neumann functions,

$\mathbf{J}_\nu(z)$: Anger functions, $\mathbf{E}_\nu(z)$: Weber functions

$\Gamma(z)$: Gamma functions.

For $z \gg 1$, we have (cf. [I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products, p. 929, p. 958])

$$\begin{aligned} J_1(z) &= \sqrt{\frac{2}{\pi z}} \left\{ [1 + R_1] \cos\left(z - \frac{3}{4}\pi\right) \right. \\ &\quad \left. - \left[\frac{1}{2z} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + R_2 \right] \sin\left(z - \frac{3}{4}\pi\right) \right\}, \quad (2.20) \end{aligned}$$

$$
\begin{aligned}
J_{-1}(z) &= \sqrt{\frac{2}{\pi z}}\left\{[1+R_1]\cos\left(z+\frac{1}{4}\pi\right)\right. \\
&\qquad \left. -\left[\frac{1}{2z}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}+R_2\right]\sin\left(z+\frac{1}{4}\pi\right)\right\}, \qquad (2.21) \\
Y_1(z) &= \sqrt{\frac{2}{\pi z}}\left\{[1+R_1]\sin\left(z-\frac{3}{4}\pi\right)\right. \\
&\qquad \left. +\left[\frac{1}{2z}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}+R_2\right]\cos\left(z-\frac{3}{4}\pi\right)\right\}, \qquad (2.22) \\
Y_{-1}(z) &= \sqrt{\frac{2}{\pi z}}\left\{[1+R_1]\sin\left(z+\frac{1}{4}\pi\right)\right. \\
&\qquad \left. +\left[\frac{1}{2z}\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}+R_2\right]\cos\left(z+\frac{1}{4}\pi\right)\right\}, \qquad (2.23)
\end{aligned}
$$

where

$$|R_1| < \left| \frac{\Gamma\left(\frac{7}{2}\right)}{8\Gamma\left(-\frac{1}{2}\right) z^2} \right|, \quad |R_2| < \left| \frac{\Gamma\left(\frac{9}{2}\right)}{48\Gamma\left(-\frac{3}{2}\right) z^3} \right|, \tag{2.24}$$

$$\mathbf{J}_{\pm 1}(z) = J_{\pm 1}(z), \tag{2.25}$$

$$\mathbf{E}_{\pm 1}(z) = -Y_{\pm 1}(z) \mp \frac{2}{\pi z^2} + O(z^{-4}). \tag{2.26}$$

Example

Example 2.2. Let

$$g_1(u) = \frac{\sin u + \sin(2u)}{4}.$$

Then $g_1(u)$ satisfies (OP). Indeed, in this case, it is clear that $a_n = 0$ $(n \in \mathbb{N}_0)$ and $b_n = 0$ $(n \geq 3)$. Therefore, we see from (3.18) that

$$\tilde{b_1} = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{4}(\cos\theta + 2\cos 2\theta)\cos\theta d\theta = \frac{1}{4}, \quad (2.27)$$

$$\tilde{b_2} = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{4}(\cos\theta + 2\cos 2\theta)\cos 2\theta d\theta = \frac{1}{2}. \quad (2.28)$$

By this, (2.17) and [I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products, pp. 30], we obtain

Example

$$
\begin{aligned}
K(\alpha) &= \alpha\left\{\tilde{b_1}V_1 + \frac{1}{2}\tilde{b_2}V_2\right\} = \frac{1}{4}\alpha(V_1+V_2) \qquad (2.29)\\
&= \frac{\alpha}{4}\left\{\sqrt{\frac{\pi}{2\alpha}}\cos\left(\alpha - \frac{3}{4}\pi\right) + \sqrt{\frac{\pi}{4\alpha}}\cos\left(2\alpha - \frac{3}{4}\pi\right) + O(\alpha^{-3/2})\right\}\\
&= \frac{\sqrt{\alpha\pi}}{8}\left\{\sqrt{2}\cos\left(\alpha - \frac{3}{4}\pi\right) + \cos\left(2\alpha - \frac{3}{4}\pi\right) + O(\alpha^{-1})\right\}.
\end{aligned}
$$

By this and (2.7), we obtain

$$
\begin{aligned}
\lambda(\alpha) &= \frac{\pi^2}{4} - \frac{\pi^{3/2}}{8\alpha^{3/2}}\left\{\sqrt{2}\cos\left(\alpha - \frac{3}{4}\pi\right) + \cos\left(2\alpha - \frac{3}{4}\pi\right)\right\}\\
&\quad + O(\alpha^{-2}).
\end{aligned}
$$

For instance, if we put $\alpha = n\pi + (3\pi)/4$ ($n \in \mathbb{N}$, $n \gg 1$), then we can easily check that $\lambda(\alpha)$ satisfies (OP).

The local behavior of $\lambda(\alpha) = \lambda(g, \alpha)$ is easy to calculate, since Taylor expansion and the time-map method work very well. We only prove Theorem 1.4 (i) for completeness.

Proof of Theorem 1.4 (i). Since $g(0) \neq 0$, by (A.1), we see that $g(0) > 0$. By (1.1), (2.5) and Taylor expansion, for $0 < u \ll 1$ and $-1 \le t \le 0$, we have

$$u_\alpha'(t) = \sqrt{\lambda(\alpha) M_\alpha(u_\alpha(t))},$$

where

$$\begin{aligned} M_\alpha(u) \quad &:= \quad 2g(0)(\alpha - u) + (1 + g'(0))(\alpha^2 - u^2) \\ &\quad + \frac{1}{3}(1 + o(1))g''(0)(\alpha^3 - u^3). \end{aligned} \tag{3.1}$$

By this and putting $s = u_\alpha(t)/\alpha$, we obtain

Local behavior of $\lambda(g, \alpha)$

$$\sqrt{\lambda(\alpha)} = \int_{-1}^{0} \frac{u'_\alpha(t)}{\sqrt{M_\alpha(u_\alpha(t))}} dt \tag{3.2}$$

$$= \sqrt{\frac{\alpha}{2g(0)}} \int_0^1 \frac{1}{\sqrt{1-s}}$$

$$\times \frac{1}{\sqrt{1 + \frac{1+g'(0)}{2g(0)}\alpha(1+s) + \frac{(1+o(1))g''(0)}{6g(0)}\alpha^2(1+s+s^2)}} ds$$

$$= \sqrt{\frac{\alpha}{2g(0)}} \int_0^1 \frac{1}{\sqrt{1-s}}$$

$$\times \left[1 - \frac{(1+g'(0))}{4g(0)}\alpha(1+s) - \frac{g''(0)}{12g(0)}\alpha^2(1+s+s^2) \right.$$

$$\left. + \frac{3}{32g(0)^2}(1+g'(0))^2\alpha^2(1+s)^2 + o(\alpha^2)\right] ds$$

Local behavior of $\lambda(g, \alpha)$

$$= \sqrt{\frac{\alpha}{2g(0)}} \left[2 - \frac{5(1 + g'(0))}{6g(0)} \alpha \right.$$
$$\left. + \left(\frac{43}{80} \frac{(1 + g'(0))^2}{g(0)^2} - \frac{11}{30} \frac{g''(0)}{g(0)} \right) \alpha^2 + o(\alpha^2) \right].$$

This implies (1.11). Thus the proof is complete. □

Bifurcation curve with nonlinear diffusion

In this section, we consider

$$
\begin{aligned}
[D(u(t))u(t)']' + \lambda f(u(t)) &= 0, \quad t \in I := (0,1), && (4.1) \\
u(t) &> 0, \quad t \in I, && (4.2) \\
u(0) &= u(1) = 0, && (4.3)
\end{aligned}
$$

where

$$
D(u) = u^k,
$$
$$
f(u) = u^{2n-k-1} + \sin u,
$$

and $\lambda > 0$ is a bifurcation parameter. Here, $n \in \mathbb{N}$ and k $(0 \le k < 2n-1)$ are constants.

Bifurcation curve with nonlinear diffusion

・(4.1)–(4.3) has been introduced by H. Lee, L. Sherbakov, J. Taber, J. Shi (2006). Especially, the case $D(u) = u^k$ $(k > 0)$ has been derived from a model equation of animal dispersal and invasion. In this situation, λ is a parameter which represents the habitat size and diffusion rate. Such model also appears as the porous media equation in material science.

・On the other hand, there are several papers which treat the asymptotic behavior of oscillatory bifurcation curves.

・Our equation (4.1)–(4.3) contains both nonlinear diffusion term and oscillatory nonlinear terms. The purpose of this talk is to find the difference between the structures of bifurcation curves of the equations with only oscillatory term and those with both nonlinear diffusion term and the oscillatory term in (4.1).

Example

To clarify our intention, let $k = 2$ and $n = 2$. Then (1.1) is given as

$$(u^2u')' + \lambda(u + \sin u) = 0. \tag{4.4}$$

The corresponding equation without nonlinear diffusion is the case $k = 0$ and $n = 1$, namely,

$$u'' + \lambda(u + \sin u) = 0. \tag{4.5}$$

・As before, by the time-map argument, for any given $\alpha > 0$, there exists a unique classical solution pair (λ, u_α) of (4.1)–(4.3) satisfying $\alpha = \|u_\alpha\|_\infty$. Furthermore, λ is parameterized by α as $\lambda = \lambda(\alpha)$ and is continuous in $\alpha > 0$. For (4.5), the following asymptotic formula for $\lambda(\alpha)$ as $\alpha \to \infty$ has been obtained.

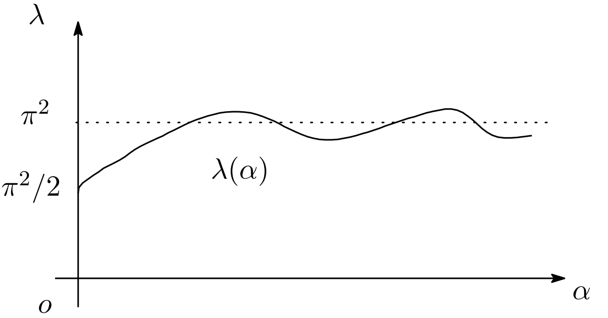
Example

Theorem 4.1. ([S, 2016]) *Consider (4.5) with (4.2)–(4.3). Then as* $\alpha \to \infty$,

$$\lambda(\alpha) \;=\; \pi^2 - 4\frac{\pi}{\alpha}\sqrt{\frac{\pi}{2\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-3/2}). \qquad (4.6)$$

For (4.5) with (4.2)–(4.3), the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \to 0$ is as follows. For a solution pair $(\lambda(\alpha), u_\alpha)$ satisfying $\|u_\alpha\|_\infty = \alpha$, put $v_\alpha(t) := u_\alpha(t)/\alpha$ and let $\alpha \to 0$. Then we easily obtain the function $v_0 \in C^2(I)$ which satisfies $-v_0''(t) = 2\lambda(0)v_0(t)$, $v_0(t) > 0$ for $t \in I$ with $v_0(0) = v_0(1) = 0$. This implies $\lambda(0) = \pi^2/2$. By this fact and Theorem 4.1, the bifurcation curve $\lambda(\alpha)$ starts from $\pi^2/2$ and tends to π^2 with oscillation and intersects the line $\lambda = \pi^2$ infinitely many times for $\alpha \gg 1$.

Rough Graph of $\lambda(\alpha)$



The graph of $\lambda(\alpha)$ for (4.6) ($k = 0, n = 1$)

Since (4.4) includes both the nonlinear diffusion function and oscillatory term, it seems interesting how the nonlinear diffusion functions give effect to the structures of bifurcation curves.

Main Theorem

Theorem 4.2. *Consider (4.1) with (4.2)–(4.3). Then as* $\alpha \to \infty$,

$$\lambda(\alpha) = 4n\alpha^{2k+2-2n} \quad (4.7)$$
$$\times \left\{ A_{k,n}^2 - 2A_{k,n}\sqrt{\frac{\pi}{2n}}\alpha^{k+(1/2)-2n} \sin\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{k+(1/2)-2n}) \right\},$$

where

$$A_{k,n} = \int_0^1 \frac{s^k}{\sqrt{1-s^{2n}}} ds. \quad (4.8)$$

Rough Graph of $\lambda(\alpha)$

By Theorem 4.2, we obtain the global behavior of $\lambda(\alpha)$ as $\alpha \to \infty$ for $n = k = 2$, and see that the asymptotic behavior of $\lambda(\alpha)$ are completely different from that for $k = 0, n = 1$ by comparing the figures.

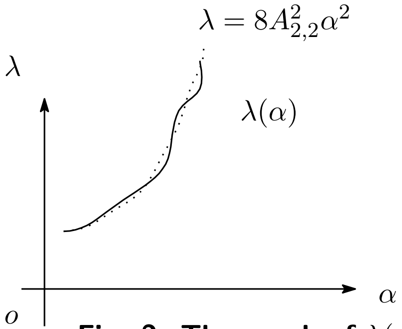


Fig. 2. The graph of $\lambda(\alpha)$ for $k = n = 2$

Now we establish the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \to 0$ to obtain a complete understanding of the structure of $\lambda(\alpha)$. Let

$$B_0 := \int_0^1 \frac{s^k}{\sqrt{1-s^{k+2}}}ds, \tag{4.9}$$

$$B_1 := \frac{k+2}{12(k+4)} \int_0^1 \frac{s^k(1-s^{k+4})}{(1-s^{k+2})^{3/2}}ds, \tag{4.10}$$

$$B_2 = \frac{k+2}{2n} \int_0^1 \frac{s^k(1-s^{2n})}{(1-s^{k+2})^{3/2}}ds, \tag{4.11}$$

$$B_3 = \frac{n}{k+2} \int_0^1 \frac{s^k(1-s^{k+2})}{(1-s^{2n})^{3/2}}ds. \tag{4.12}$$

Theorem 4.3. *Consider (4.1) with (4.2)–(4.3). Then the following asymptotic formulas hold as* $\alpha \to 0$.

(i) *Assume that* $k + 4 < 2n$. *Then*

$$\lambda(\alpha) = 2(k+2)\alpha^k \{B_0^2 + 2B_0 B_1 \alpha^2 + o(\alpha^2)\}. \tag{4.13}$$

(ii) *Assume that* $2n = k + 4$. *Then*

$$\lambda(\alpha) = 2(k+2)\alpha^k \{B_0^2 - 10B_0 B_1 \alpha^2 + o(\alpha^2)\}. \tag{4.14}$$

(iii) *Assume that* $k + 2 < 2n < k + 4$. *Then*

$$\lambda(\alpha) = 2(k+2)\alpha^k \{B_0^2 - B_0 B_2 \alpha^{2n-k-2} + o(\alpha^{2n-k-2})\}. \tag{4.15}$$

(iv) *Assume that* $2n = k + 2$. *Then*

$$\lambda(\alpha) = (k+2)\alpha^k\{B_0^2 + B_0B_1\alpha^2 + o(\alpha^2)\}. \quad (4.16)$$

(v) *Assume that* $k + 1 < 2n < k + 2$. *Then*

$$\lambda(\alpha) = 4n\alpha^{2(k+1-n)}\{A_{k,n}^2 - 2A_{k,n}B_3\alpha^{k+2-2n} + o(\alpha^{k+2-2n})\}. \quad (4.17)$$

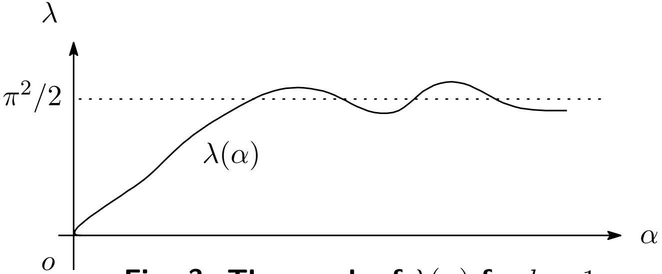


Fig. 3. The graph of $\lambda(\alpha)$ for $k = 1$, $n = 2$

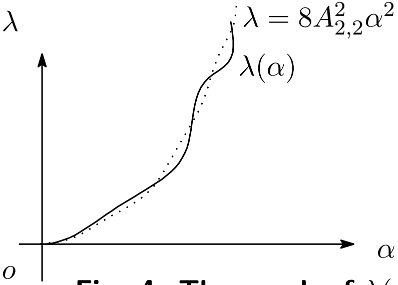


Fig. 4. The graph of $\lambda(\alpha)$ for $k = n = 2$

Rough Shape of $\lambda(\alpha)$

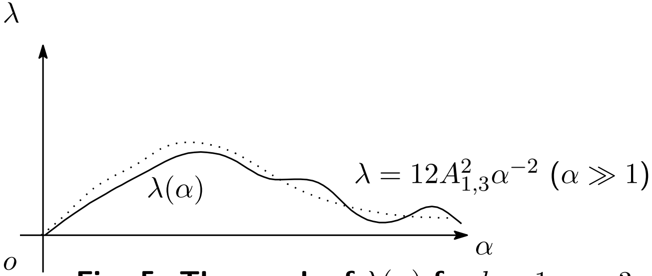


Fig. 5. The graph of $\lambda(\alpha)$ for $k = 1, n = 3$

Sketch of the proofs

The proofs depend on the generalized time-map argument and stationary phase method (cf. Lemma 5.1 below). It should be mentioned that, if we apply Lemma 5.1 to our situation, careful consideration about the regularity of the functions is necessary.

By the generalized time-map obtained in [16] (cf. (5.7) below) and the standard time-map argument, we see that for any given $\alpha > 0$, there exists a unique classical solution pair (λ, u_α) of (4.1)–(4.3) satisfying $\alpha = \|u_\alpha\|_\infty$. Furthermore, λ is parameterized by α as $\lambda = \lambda(\alpha)$ and is continuous in $\alpha > 0$.

For $u \geq 0$, we put

$$G(u) := \int_0^u f(y)D(y)dy = \frac{1}{2n}u^{2n} + G_1(u) \tag{5.1}$$
$$:= \frac{1}{2n}u^{2n} + \int_0^u y^k \sin y dy.$$

It is known that if $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (4.1)–(4.3), then

$$u_\alpha(t) = u_\alpha(1-t), \quad 0 \leq t \leq 1, \tag{5.2}$$
$$u_\alpha\left(\frac{1}{2}\right) = \max_{0\leq t\leq 1} u_\alpha(t) = \alpha, \tag{5.3}$$
$$u'_\alpha(t) > 0, \quad 0 < t < \frac{1}{2}. \tag{5.4}$$

For $0 \leq s \leq 1$ and $\alpha \gg 1$, we have

$$\left|\frac{G_1(\alpha) - G_1(\alpha s)}{\alpha^{2n}(1-s^{2n})}\right| = \left|\frac{\int_{\alpha s}^{\alpha} w^k \sin w dw}{\alpha^{2n}(1-s^{2n})}\right| \leq C\alpha^{k+1-2n} \ll 1. \tag{5.5}$$

Fundanental tools

By (4.1), we have

$$\{[D(u(t))u(t)']' + \lambda f(u(t))\}[D(u(t))u(t)'] = 0.$$

Namely, by putting $t = 1/2$,

$$\frac{d}{dt}\left[\frac{1}{2}[D(u(t))u(t)']^2 + \lambda G(u(t))\right] = 0,$$

$$\frac{1}{2}[D(u(t))u(t)']^2 + \lambda G(u(t)) = \text{const.} = \lambda G(\alpha),$$

$$D(u(t))u(t)' = \sqrt{2\lambda(G(\alpha) - G(u(t))}, \quad 0 \le t \le \frac{1}{2}.$$

By putting $u = u(t)$,

$$\frac{1}{2}\sqrt{2\lambda} \;\; = \;\; \int_0^{1/2} \frac{D(u(t)u'(t)}{\sqrt{2\lambda(G(\alpha) - G(u(t))}} dt = \int_0^{\alpha} \frac{D(u)}{\sqrt{G(\alpha) - G(u)}} du.$$

By this, (5.2) and Taylor expansion, we see that

$$\sqrt{\frac{\lambda(\alpha)}{2}} = \int_0^\alpha \frac{D(u)}{\sqrt{G(\alpha)-G(u)}}du \tag{5.6}$$

$$= \int_0^\alpha \frac{u^k}{\sqrt{\frac{1}{2n}(\alpha^{2n}-u^{2n})+G_1(\alpha)-G_1(u)}}du$$

$$= \sqrt{2n}\alpha^{k+1-n}\int_0^1 \frac{s^k}{\sqrt{1-s^{2n}+\frac{2n}{\alpha^{2n}}(G_1(\alpha)-G_1(\alpha s))}}ds$$

$$= \sqrt{2n}\alpha^{k+1-n}\int_0^1 \frac{s^k}{\sqrt{1-s^{2n}}}\left\{1-\frac{n}{\alpha^{2n}}\frac{G_1(\alpha)-G_(\alpha s)}{(1-s^{2n})}(1+o(1))\right\}ds$$

$$= \sqrt{2n}\alpha^{k+1-n}\left\{\int_0^1 \frac{s^k}{\sqrt{1-s^{2n}}}ds - \frac{n}{\alpha^{2n}}L(\alpha)(1+o(1))\right\},$$

where

$$L(\alpha) := \int_0^1 \frac{s^k}{(1-s^{2n})^{3/2}}(G_1(\alpha) - G_1(\alpha s))ds. \qquad (5.7)$$

We see from (5.6) and (5.7) that if we obtain the precise asymptotic formula for $L(\alpha)$ as $\alpha \to \infty$, then we obtain Theorem 4.2. To do this, we apply the stationary phase method to our situation. Indeed, we have the following equality.

Lemma 5.1. *Assume that the function* $f(r) \in C^2[0,1]$, $w(r) \in C^3[0,1]$ *and*

$$w'(r) < 0, \quad r \in (0,1], \quad w'(0) = 0, \quad w''(0) < 0. \tag{5.8}$$

Then as $\mu \to \infty$

$$\int_0^1 f(r) e^{i\mu w(r)} dr = \frac{1}{2} e^{i(\mu w(0) - (\pi/4))} \sqrt{\frac{2\pi}{\mu |w''(0)|}} f(0) + O\left(\frac{1}{\mu}\right). \tag{5.9}$$

In particular, by taking the imaginary part of (5.9), as $\mu \to \infty$,

$$\int_0^1 f(r) \sin(\mu w(r)) dr = \frac{1}{2} \sqrt{\frac{2\pi}{\mu |w''(0)|}} f(0) \sin\left(w(0)\mu - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu}\right) \tag{5.10}$$

Key Lemma

Lemma 5.2. *As* $\alpha \to \infty$,

$$L(\alpha) = \sqrt{\frac{\pi}{2}} \frac{1}{n^{3/2}} \alpha^{k+(1/2)} \sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^k). \tag{5.11}$$

Proof. We put $s = \sin\theta$ and

$$
\begin{aligned}
Y(\theta) &:= Y_1(\theta)(G_1(\alpha) - G_1(\alpha\sin\theta)) \qquad (5.12) \\
&:= \frac{\sin^k\theta}{(1+\sin^2\theta+\cdots+\sin^{2n-2}\theta)^{3/2}}(G_1(\alpha)-G_1(\alpha\sin\theta)).
\end{aligned}
$$

By integration by parts, we have

$$
\begin{aligned}
L(\alpha) &= \int_0^1 \frac{s^k(G_1(\alpha)-G_1(\alpha s))}{(1-s^2)^{3/2}(1+s^2+\cdots+s^{2n-2})^{3/2}}ds \qquad (5.13) \\
&= \int_0^{\pi/2} \frac{1}{\cos^2\theta}\frac{\sin^k\theta(G_1(\alpha)-G_1(\alpha\sin\theta))}{(1+\sin^2\theta+\cdots+\sin^{2n-2}\theta)^{3/2}}d\theta \\
&:= L_1(\alpha) - L_2(\alpha) \\
&= [\tan\theta Y(\theta)]_0^{\pi/2} - \int_0^{\pi/2}\tan\theta\{Y_1(\theta)(G_1(\alpha)-G_1(\alpha\sin\theta))\}'d\theta.
\end{aligned}
$$

Proof of Key Lemma

By l'Hôpital's rule, we obtain

$$\lim_{\theta\to\pi/2}\frac{G_1(\alpha)-G_1(\alpha\sin\theta)}{\cos\theta} \tag{5.14}$$
$$=\lim_{\theta\to\pi/2}\frac{\alpha\cos\theta(\alpha\sin\theta)^k\sin(\alpha\sin\theta)}{\sin\theta}=0.$$

This implies that $L_1(\alpha)=0$. Next,

$$\begin{aligned} L_2(\alpha) &= \int_0^{\pi/2}\tan\theta\{Y_1'(\theta)(G_1(\alpha)-G_1(\alpha\sin\theta)\} \\ &\qquad - Y_1(\theta)\alpha\cos\theta(\alpha\sin\theta)^k\sin(\alpha\sin\theta)\}d\theta. \\ &:= L_{21}(\alpha)-L_{22}(\alpha). \end{aligned} \tag{5.15}$$

We first calculate $L_{21}(\alpha)$. Assume that $k > 0$. Then

$$Y_1'(\theta) = \frac{\sin^{k-1}\theta\cos\theta}{(1+\sin^2\theta+\cdots+\sin^{2n-2}\theta)^{3/2}} \times \left[k - \frac{3(\sin^2\theta + 2\sin^4\theta + \cdots + (n-1)\sin^{2n-2}\theta)}{1+\sin^2\theta+\cdots+\sin^{2n-2}\theta}\right]. \tag{5.16}$$

This implies that for $\alpha \gg 1$,

$$|\tan\theta Y_1'(\theta)| \le C|\sin^k\theta| \le C. \tag{5.17}$$

By direct calculation, we also obtain (5.17) for the case where $k = 0$. By integration by parts, we obtain

Proof of Key Lemma

$$\begin{aligned} |G_1(\alpha) - G_1(\alpha\sin\theta)| &= \left| \int_{\alpha\sin\theta}^{\alpha} w^k \sin w dw \right| \\ &\le \left| \left[-w^k \cos w \right]_{\alpha\sin\theta}^{\alpha} \right| + \left| \int_{\alpha\sin\theta}^{\alpha} k w^{k-1} \cos w dw \right| \\ &\le C\alpha^k. \end{aligned} \quad (5.18)$$

By (5.17) and (5.18), for $\alpha \gg 1$, we obtain

$$|L_{21}(\alpha)| = |\tan\theta Y_1'(\theta)(G_1(\alpha) - G_1(\alpha\sin\theta))| \le C\alpha^k. \quad (5.19)$$

Since

$$L_{22}(\alpha) = \alpha^{k+1} \int_0^{\pi/2} Y_1(\alpha) \sin^{k+1}\theta \sin(\alpha \sin\theta) d\theta, \qquad (5.20)$$

by putting $\theta = \dfrac{\pi}{2}(1-r)$, we obtain

$$L_{22}(\alpha) = \frac{\pi}{2}\alpha^{k+1} \int_0^1 \frac{\cos^{2k+1} \frac{\pi}{2}r}{(1+\cos^2 \frac{\pi}{2}r + \cdots + \cos^{2n-2} \frac{\pi}{2}r)^{3/2}} \times \sin\left(\alpha \cos \frac{\pi}{2}r\right) dr. \qquad (5.21)$$

Let

$$f(r) = \frac{\cos^{2k+1} \frac{\pi}{2}r}{(1+\cos^2 \frac{\pi}{2}r + \cdots + \cos^{2n-2} \frac{\pi}{2}r)^{3/2}}, w(r) = \cos\frac{\pi}{2}r, \mu = \alpha. \qquad (5.22)$$

Proof of Key Lemma

Case 1. Assume that $k > 1/2$ or $k = 0$. Then clearly $f(r) \in C^2[0,1]$, and we are able to apply Lemma 5.1 to (5.21). Then we obtain

$$L_{22}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{1}{n^{3/2}} \alpha^{k+(1/2)} \sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^k). \tag{5.23}$$

By this, (5.16) and (5.19), we obtain (5.11).

Case 2. Assume that $0 < k < 1/2$. Then $f(r) \in C^{1+2k}[0,1]$ with $0 < 2k < 1$. Therefore, $f(r)$ does not satisfy the condition in Lemma 5.1. However, by the modefication of the argument, we find that we can still apply Lemma 5.1 to (5.21) in this situation and obtain (5.23). Thus the proof is complete. □

By (5.6) and Lemma 5.2, we obtain Theorem 4.2 immediately. Thus the proof is complete. □

In this section, let $0 < \alpha \ll 1$. The proofs of Theorem 4.3 (i)-(v) are similar. Therefore, we only prove (i).

Proof of Theorem 4.3 (i). We assume that $2n > k+4$. Then by Taylor expansion, for $0 \le s \le 1$, we have

$$G(\alpha) - G(\alpha s) = \frac{1}{2n}\alpha^{2n}(1 - s^{2n}) + \frac{1}{k+2}\alpha^{k+2}(1 - s^{k+2}) - \frac{1}{6(k+4)}\alpha^{k+4}(1 - s^{k+4})(1 + o(1)). \quad (6.1)$$

By this, Taylor expansion and putting $u = \alpha s$, we obtain

$$\sqrt{\frac{\lambda(\alpha)}{2}} = \int_0^{\alpha}$$
$$\frac{u^k du}{\sqrt{\frac{1}{2n}(\alpha^{2n} - u^{2n}) + \frac{1}{k+2}(\alpha^{k+2} - u^{k+2}) - \frac{1}{6(k+4)}(\alpha^{k+4} - u^{k+4})(1 + o(1))}}$$
$$= \sqrt{k+2}\alpha^{k/2} \int_0^1 \frac{s^k}{\sqrt{1 - s^{k+2}}\sqrt{1 - \frac{k+2}{6(k+4)}\frac{1-s^{k+4}}{1-s^{k+2}}\alpha^2 + o(\alpha^2)}} ds$$
$$= \sqrt{k+2}\alpha^{k/2} \int_0^1 \frac{s^k}{\sqrt{1 - s^{k+2}}} \left(1 + \frac{k+2}{12(k+4)}\frac{1 - s^{k+4}}{1 - s^{k+2}}\alpha^2 + o(\alpha^2)\right) ds$$
$$= \sqrt{k+2}\alpha^{k/2}\{B_0 + B_1\alpha^2 + o(\alpha^2)\}. \quad (6.2)$$

This implies (4.13). Thus the proof is complete. □

In this section, we show that Case 2 in Lemma 5.2 holds for completeness. We put

$$f(x) = f_1(x)f_2(x) := \cos^{2k+1}\frac{\pi}{2}x \, \frac{1}{(1+\cos^2\frac{\pi}{2}x + \cdots + \cos^{2n-2}\frac{\pi}{2}x)^{3/2}} \quad (7.1)$$

Note that $0 < 2k < 1$. We see that $f_2(x) \in C^2[0,1]$. The essential point of the proof of Lemma 5.1 in this case is to show [Korman, (2012), Lemmas 2.24 and 2.25] holds with less regularity. Namely, as $\mu \to \infty$,

$$\Phi(\mu) := \int_0^1 f(x)e^{-i\mu x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i(\pi/4)}f(0) + O\left(\frac{1}{\mu}\right). \quad (7.2)$$

We put $h(x) = (f(x) - f(0))/x$. Then we have $f(x) = f(0) + xh(x)$. We know from [Korman (2012), Lemmas 2.24] that for $\mu \gg 1$,

$$\int_0^1 e^{-i\mu x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i\pi/4} + O\left(\frac{1}{\mu}\right). \quad (7.3)$$

By (7.2) and (7.3), we obtain

$$\begin{aligned}\Phi(\mu) &= f(0)\int_0^1 e^{-i\mu x^2}dx + \int_0^1 xe^{-i\mu x^2}h(x)dx \\ &= \frac{1}{2}f(0)\sqrt{\frac{\pi}{\mu}}e^{-i\pi/4} + O\left(\frac{1}{\mu}\right) + \int_0^1 xe^{-i\mu x^2}h(x)dx.\end{aligned} \tag{7.4}$$

We put

$$\Phi_1(\mu) := \int_0^1 xe^{-i\mu x^2}h(x)dx. \tag{7.5}$$

Now we prove that $h(x) \in C^1[0,1]$, because if it is proved, then by integration by parts, we easily show that $\Phi_1(\mu) = O(1/\mu)$ and our conclusion (7.2) follows immediately from (7.4) and (7.5). For $0 \le x \le 1$, we have

$$
\begin{aligned}
h(x) &= \frac{f(x)-f(0)}{x} \\
&= f_2(x)\frac{f_1(x)-f_1(0)}{x} + f_1(0)\frac{f_2(x)-f_2(0)}{x} \\
&:= f_2(x)h_1(x) + f_1(0)h_2(x).
\end{aligned} \tag{7.6}
$$

Then we have $h_2(x) \in C^1[0,1]$. Furthermore, by direct calculation, we can show that $h_1(x) \in C^1[0,1]$. It is reasonable, because by Taylor expansion, for $0 < x \ll 1$, we have

$$
h_1(x) = -\frac{(2k+1)\pi^2}{8}x + O(x^3). \tag{7.7}
$$

Thus the proof is complete. □

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Thank you very much

Thank You for Your Attention