放物型偏微分方程式における動的特異点

柳田英二（東京工業大学）

with S. Sato, M. Hoshino, J. Takahashi, K. P. P. Htoo,
M. Fila, T. Kan, C. Bandle, J.-L. Chern, G. Hwang

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Plan of my lectues:

Part I: Introduction

Part II: Linear equations

Heat equation: $u_t = \Delta u$

Dynamic Hardy potential: $u_t = \Delta u + \frac{\lambda(t)}{|x - \xi(t)|^2} u$

Part III: Nonlinear equations

Nonlinear diffusion: $u_t = \Delta u^m$

Absorption equation: $u_t = \Delta u - u^p$

Fujita equation: $u_t = \Delta u + u^p$

Part IV: Related topics

Part I: Introduction

Singularity:

A point at which a given mathematical object is not defined or not well-behaved (e.g., infinite or not differentiable).

- **Gravitational theory, Material science, Meteorology**
- **Algebraic geometry**
  Singular point of an algebraic variety:
  A point where an algebraic variety is not locally flat.
- **Differential geometry**
  Singular point of a manifold:
  A point where the manifold is not given by a smooth embedding of a parameter.
- **Complex analysis**
  Poles and essential singularities.

Is it good news or bad news to encounter singularities?

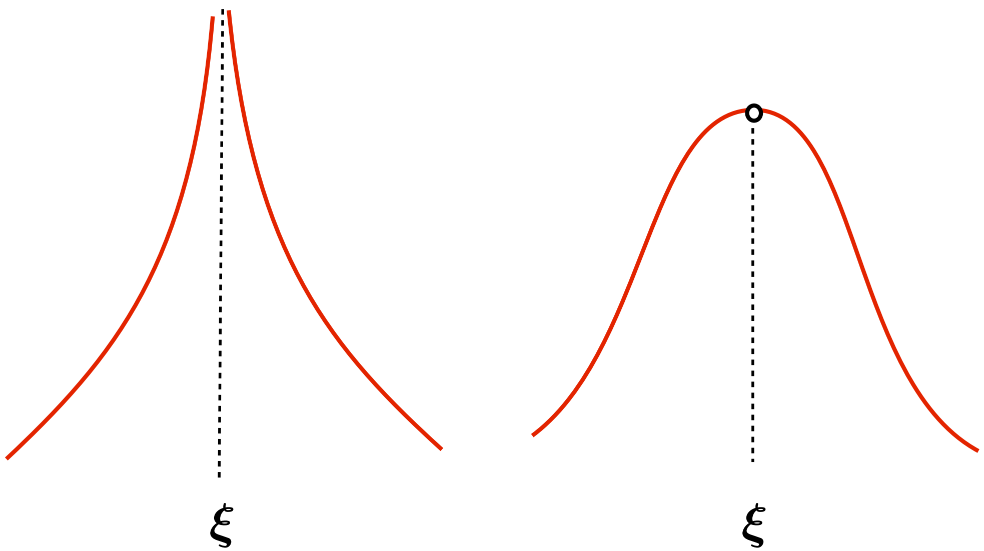
Riemann's Removability Theorem (1851)

Let $f(z)$ be any holomorphic function on a punctured domain $\Omega \setminus \{\xi\} \subset \mathbb{C}$. The singularity ξ is removable (i.e., $f(z)$ is holomorphically extendable to Ω) if and only if

$$f(z) = o(|z - \xi|^{-1}) \quad (z \to \xi).$$

In fact, he classified all possible isolated singularities :

- **Removable singularities.**
- **Poles of order $n = 1, 2, \ldots$**
- **Essential singularities.**



$$f(z) = \frac{1}{(z-\xi)^n} \qquad (n \in \mathbb{N}) \qquad \cdots \text{ pole, non-removable.}$$

$$f(z) = \frac{\sin(z-\xi)}{z-\xi} \qquad \cdots \text{ removable by setting } f(\xi) = 1.$$

Any non-removable singularity must be a pole of order $\exists n \geq 1$ or an essential singularity.

There have been many studies on singularities in linear and nonlinear elliptic PDEs.

Laplace equation:

$$\Delta u = 0 \qquad \text{on } \Omega \setminus \{\xi\},$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$.

- **The singularity is removable if and only if**

$$u(x) = o(|x-\xi|^{-(N-2)}) \qquad (x \to \xi).$$

... Weyl (1940)

- **The fundamental solution**

$$u(x) = C_N |x-\xi|^{-(N-2)}$$

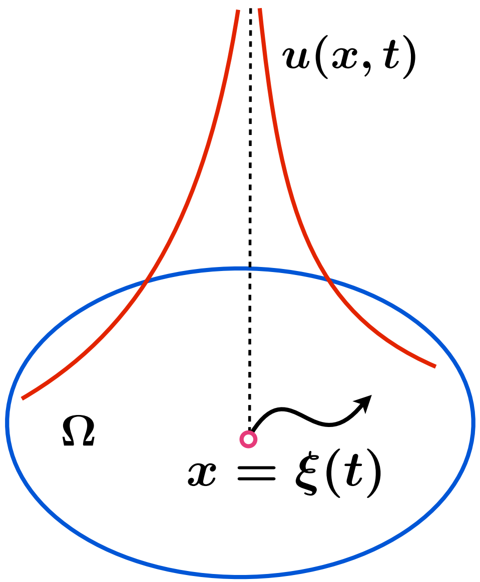
has a non-removable singularity.

Other results on the removability of singularities:

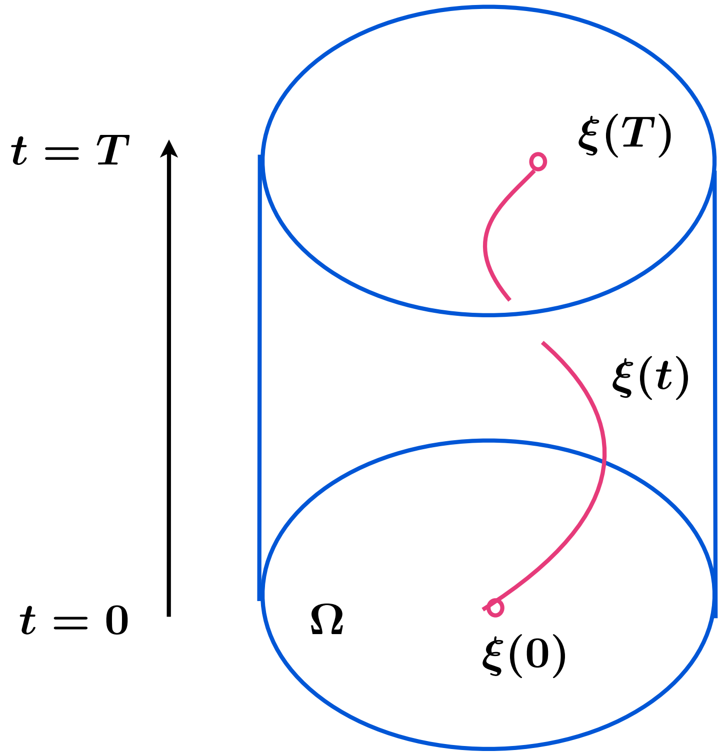
- **Heat equation: Weyl (1940)**
- **Harmonic maps: Sacks-Uhlenbeck (1981)**
- **Nonlinear parabolic equation: Brezis-Friedman (1983)**
- $\cdots\cdots$

Question

For parabolic PDEs, what if a singularity $\xi = \xi(t)$ is moving?



Moving singularity



Time-space domain

$$D = \{(x,t) : x \in \Omega \setminus \{\xi(t),\ 0 \le t \le T\}$$

Target: Moving singularities

- Removability
- Local and global existence
- Non-existence
- Asymptotic profile
- Uniqueness and Classification

Part II: Linear equations

[Heat equation with a moving singularity]

... with Jin Takahashi, Khin Phyu Phyu Htoo, Toru Kan

$$u_t = \Delta u, \qquad x \in \mathbb{R}^N \setminus \{\xi(t)\}, \quad t \in (0, T).$$

Basic assumptions:

- $N \geq 3$ mostly, $N = 2$ ot $N = 1$ occasionally.
- Consider nonnegative solutions only.
- $u(x, t)$ satisfies the equation in the classical sense for $x \neq \xi(t)$.
- $\xi(t)$ is continuous.

Standing singularity $\xi(t) \equiv \xi_0$

$$u_t = \Delta u \quad \text{in } \Omega \setminus \{\xi_0\}, \quad t \in (0,T).$$

- **Non-removable singularity**

  There exists a solution with a singularity

$$u(x,t) = \begin{cases} |x-\xi_0|^{-N+2} + \cdots & \text{for } N \geq 3, \\ \log(|x-\xi_0|^{-1}) + \cdots & \text{for } N = 2, \end{cases}$$

- **Removability ... Hsu (2010), Hui (2010)**

  For $N \geq 3$, the singular point ξ_0 is removable if and only if

$$|u(x,t)| = o(|x-\xi_0|^{-N+2}) \quad \text{as} \quad x \to \xi_0$$

uniformly in $t \in (0,T)$.

Removability of a moving singularity

$$u_t = \Delta u, \qquad x \neq \xi(t), \quad t \in (0, T).$$

Theorem (Removability)

Suppose that $\xi(t)$ is locally at least 1/2−Hölder continuous in $t \in [0, T]$. Then the singularity is removable if and only if

$$u(x, t) = o(|x - \xi(t)|^{-(N-2)})$$

uniformly in $t \in (0, T)$.

1/2-Hölder continuity is essential.
Brownian motion is $(1/2 - \varepsilon)$−Hölder continuous in t.

Proof. By assumption

$$|u(x,t)| = o(|x-\xi|^{-(N-2)}) \qquad (x \to \xi)$$

and 1/2−Hölder continuity, for any $0 < t_1 < t_2 < T$, the solution u satisfies

$$\int_\Omega u(x,t_2)\phi(x,t_2) - u(x,t_1)\phi(x,t_1)\,dx = \int_{t_1}^{t_2}\int_\Omega u(\phi_t + \Delta\phi)\,dxdt$$

for all $\phi \in C_0^\infty(\Omega \times (0,T))$. Here, 1/2−Hölder continuity is necessary for the construction of suitable cut-off functions around the curve $x = \xi(t)$. Hence $u \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$ satisfies the heat equation in $\Omega \times (0,T)$ in the distribution sense.

Then by the Weyl lemma, u satisfies the heat equation in $\overline{\Omega} \times (0,T)$ in the classical sense.

Remark. For $N = 2$, the singularity of u at $x = \xi(t)$ is removable if and only if

$$u(x,t) = o(\log|x - \xi(t)|^{-1})$$

uniformly in $t \in (0,T)$.

Remark. For $N = 1$, if we define $\tilde{u}$ by

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & \text{for } x \neq \xi(t) \\ \liminf_{x \uparrow \xi(t)} u(x,t) & \text{for } x = \xi(t) \end{cases}$$

then the singularity at $x = \xi(t)$ is removable if and only if $\tilde{u}$ is continuously differentiable at $x = \xi(t)$ for any $t \in (0,T)$.

Non-removable singularity

Theorem (Existence of a moving singularity)

Let $N > 2$, $T > 0$. Given any $\xi(t) : [0, T] \to \mathbb{R}^N$ and any positive continuous function $a(t)$, there exists a solution with a singularity

$$u(x,t) \simeq a(t)|x - \xi(t)|^{-N+2}$$

at $x = \xi(t)$.

$$\begin{cases} u_t = \Delta u + g(x,t), & x \in \mathbb{R}^N, \; t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

Representation formula of the solution:

$$u(x,t) := \int_{\mathbb{R}^N} G(x,y,s)u_0(y)dy + \int_0^s \int_{\mathbb{R}^N} G(x,y,s)g(y,s)dyds,$$

where

$$G(x,y,t) = \frac{1}{(4\pi t)^{-N/2}} \exp\Big(- \frac{|x-y|^2}{4t} \Big).$$

Proof. Consider the initial value problem

$$\begin{cases} u_t - \Delta u = C_N a(t)\delta(x - \xi(t)), & x \in \mathbb{R}^N, \ t > 0 \\ u(x,0) = 0, & x \in \mathbb{R}^N, \end{cases}$$

where $\delta(\cdot)$ denotes the Dirac delta. Then

$$u(x,t) = \int_0^t \frac{a(s)}{(4\pi(t-s))^{N/2}} \exp\Big(- \frac{|x - \xi(s)|^2}{4(t-s)}\Big)\, ds$$

is the desired singular solution.

This is intuitively clear from the representation formula

$$u(x,t) = \int_0^t \int_{\mathbb{R}^N} G(x,y,s)a(s)\delta(y - \xi(s))dyds = \int_0^t G(x,\xi(s),s)a(s)ds$$

but this needs a proof. □

Key observation

$$\exp\Big(-\frac{|x-\xi(s)|^2}{4(t-s)}\Big) = \exp\Big(-\frac{|x-\xi(t)+\xi(t)-\xi(s)|^2}{4(t-s)}\Big)$$
$$\simeq \exp\Big(-\frac{|x-\xi(t)|^2}{4(t-s)}\Big)\cdot\Big(-\exp\Big(-\frac{|t-s|^{2\gamma}}{4(t-s)}\Big)$$
$$\simeq \exp\Big(-\frac{|x-\xi(t)|^2}{4(t-s)}\Big)$$

Related results

- If $\xi(t)$ has less regularity, anomalous singularities may appear. In fact, the singularity could be weaker and asymmetric.

  ... Kan-Takahashi (2016)

- More general inhomogeneous term

  $$u_t - \Delta u = g(x,t): \text{ Radon measure}$$

  .... Kan-Takahashi (2016,2017)

- Higher dimensional singular set with the codimension 3 or higher.

  ... Htoo-Takahashi-Y (2018)

[Dynamic Hardy potential]

with Jann-Long Chern, Jin Takahashi, Gyeongha Hwang

Parabolic equation with a Hardy potential

$$u_t = \Delta u + \frac{\lambda}{|x-\xi_0|^2}u, \quad x \in \mathbb{R}^N \setminus \{\xi_0\},$$

where $N \geq 3$. Baras-Goldstein (1984) showed that

$$\lambda_c := \frac{(N-2)^2}{4} > 0$$

is critical in the following sense:

(i) if $0 < \lambda \leq \lambda_c$, there exists a global solution satisfying

$$u(x,t) \geq c|x-\xi_0|^{-\alpha_1}, \qquad |x-\xi_0| < 1.$$

(ii) If $\lambda > \lambda_c$, then there exist no positive solutions.

Steady state

$$\Delta u + \frac{\lambda}{|x-\xi_0|^2} u, \qquad x \in \mathbb{R}^N.$$

Substituting $u = r^{-\alpha}$, $r = |x - \xi_0|$, **then**

$$u_{rr} + \frac{N-1}{r} u_r + \frac{\lambda}{r^2} = \{\alpha(\alpha-1) + (N-1)\alpha + \lambda\} r^{-\alpha-2} = 0.$$

Hence α **must satisfy**

$$\alpha^2 - (N-2)\alpha + \lambda = 0.$$

If $\lambda < \lambda_c = (N-2)^2/4$, **the quadratic equation has two real roots** $0 < \alpha_1 < \alpha_2 < N-2$:

$$\begin{aligned} 0 < \alpha_1 &= \frac{N-2-\sqrt{(N-2)^2-4\lambda}}{2} < \frac{N-2}{2} \\ < \alpha_2 &= \frac{N-2+\sqrt{(N-2)^2-4\lambda}}{2} < N-2 \end{aligned}$$

Heat equation with a dynamic Hardy term

$$u_t = \Delta u + V(x,t)u, \quad x \in \mathbb{R}^N \setminus \{\xi(t)\}.$$

Assumptions

- $V(x,t)$ is positive and continuous in $(x,t) \in \mathbb{R}^N \setminus \{\xi(t)\} \times [0,T]$, and is bounded for $|x| > 1$.
- $V(x,t)$ is singular at $\xi(t)$:

$$V(x,t) = \lambda(t)|x-\xi(t)|^{-2} + O(|x-\xi(t)|^{-2+\varepsilon}) \quad (x \to \xi(t)),$$

- $\xi = \xi(t)$ is γ-Hölder continuous with $\gamma > 1/2$.
- $\lambda(t)$ is a smooth positive function of $t \in [0,T]$.

Example

$$u_t = \Delta u + \frac{\lambda(t)}{|x-\xi(t)|^2}u, \quad x \in \mathbb{R}^N \setminus \{\xi(t)\}.$$

Minimal solution

Define

$$V_n(x,t) := \min\{V(x,t), n\}.$$

If $u_0 \in L^1(\mathbb{R}^N)$, then for each $n \in \mathbb{N}$, there exists a unique bounded solution of

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + V_n(x,t)u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

We denote the unique solution by $u_n(x,t)$. In this case, by the comparison theorem, $\{u_n(x,t)\}$ is a monotone increasing sequenced. Hence if $\{u_n(x,t)\}$ is bounded, then

$$u(x,t) := \lim_{n\to\infty} u_n(x,t)$$

exists. Then the parabolic regularity implies that the limiting function $u(x,t)$ satisfies the equation for $x \neq \xi(t)$. We call such $u(x,t)$ a minimal solution. For the existence of a minimal solution, it suffices to find a supersolution.

Theorem (Existence of a minimal solution)

Assume

$$0 < V(x,t) \le \frac{\lambda}{|x - \xi(t)|^2}, \qquad 0 < |x - \xi(t)| < 1,$$

for some $0 < \lambda < \lambda_c$. If the initial value satisfies

$$u_0(x) \le C|x - \xi(0)|^{-k}, \qquad k < \alpha_2(\lambda) + 2 = N - \alpha_1(\lambda),$$

then there exists a minimal solution satisfying

$$u(x,t) \le C|x - \xi(t)|^{-\alpha_1(\lambda)}.$$

Idea of the proof

STEP 1: Existence in the simplest case (standing singularity).
STEP 2: Comparison with a moving singularity.
STEP 3: Gronwall-like argument.

STEP 1: Existence in the simplest case

$$u_t^+ = \Delta u^+ + \frac{\lambda}{|x|^2} u^+, \qquad x \neq 0.$$

Radial solution $u = v(r)$, $r = |x|$, satisfies

$$\begin{cases} v_t = v_{rr} + \dfrac{N-1}{r} v_r + \dfrac{\lambda}{r^2} v, & r > 0, \quad t > 0, \\ v(r,0) = v_0(r), & r > 0, \end{cases}$$

where $v_0(r)$ is continuous in $r > 0$. Setting $w(r,t) := r^{\alpha_1} v(r,t)$, we obtain the radial heat equation

$$w_t = w_{rr} + \frac{d-1}{r} w_r, \qquad r > 0,$$

where $d = N - 2\alpha_1 > 2$.

Forward self-similar solution $w = t^{-l}\varphi(\rho)$, $\rho = t^{-\frac{1}{2}}r$, must satisfy

$$\varphi_{\rho\rho} + \frac{d-1}{\rho}\varphi_\rho + \frac{\rho}{2}\varphi_\rho + l\varphi = 0, \quad \rho > 0.$$

Lemma (Haraux-Weissler equation)

If $l < d/2$, then the solution with $\varphi(0) = 1$ remains positive for all $\rho > 0$. Moreover, there exists a constant $c(l) > 0$ such that

$$\varphi(\rho) = c(l)\rho^{-2l} + o(\rho^{-2l}) \quad \text{as } \rho \to \infty,$$

Lemma (Radial singular solution)

If $u_0(x) = |x|^{-k}$ with $k < \alpha_2 + 2$, there exists a minimal solution given by

$$u(x,t) = \frac{1}{c(l)} t^{-l} |x|^{-\alpha_1} \varphi(t^{-1/2}|x|),$$

where $l = (k - \alpha_1)/2$.

Hence there exists a minimal solution in the simple case.

STEP 2: Comparison with the moving singularity

Consider the equations

$$u_t = \Delta u + V_n(x - \xi_0)u, \qquad x \neq 0,$$

and

$$\tilde{u}_t = \Delta \tilde{u} + V_n(x - \xi(t))\tilde{u}, \qquad x \neq 0,$$

with the same initial value

$$u(x,0) = \tilde{u}(x,0) = u_0(x) \ (= |x - \xi_0|^{-k}),$$

where $\xi_0 = \xi(0)$. We shall estimate the difference

$$w(x,t) := \tilde{u}(x - \xi(t), t) - u(x - \xi_0, t).$$

Since $\xi(t)$ may **NOT** be differentiable, we use the integral formulas

$$u(x,t) = \int_{\mathbb{R}^N} G(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)V_n(y)u(y,s)dyds,$$

$$\tilde{u}(x,t) = \int_{\mathbb{R}^N} G(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)V_n(y)\tilde{u}(y,s)dyds.$$

By the change of variables, we have

$$\begin{aligned} u(x+\xi_0,t) &= \int_{\mathbb{R}^N} G(x-y,t)u_0(y+\xi_0)dy \\ &\quad + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)V_n(y)u(y+\xi_0,s)dyds, \end{aligned}$$

$$\begin{aligned} \tilde{u}(x+\xi(t),t) &= \int_{\mathbb{R}^N} G(x-y+\xi(t)-\xi_0),t)u_0(y+\xi_0)dy \\ &\quad + \int_0^t \int_{\mathbb{R}^N} G(x-y+\xi(t)-\xi(s),t-s)V_n(y)\tilde{u}(y+\xi(s),s)dyds. \end{aligned}$$

Hence $w(x,t) := \tilde{u}(x+\xi(t),t) - u(x+\xi_0,t)$ **satisfies**

$$\begin{aligned} w(x,t) &= \int_{\mathbb{R}^N} \{G(x-y+\xi(t)-\xi_0,t) - G(x-y,t)\}u_0(y+\xi_0)\}dy \\ &\quad + \int_0^t \int_{\mathbb{R}^N} G(x-y+\xi(t)-\xi(s),t-s)V_n(y)w(y+\xi(s),s)dyds \\ &\quad + \int_0^t \int_{\mathbb{R}^N} \{G(x-y+\xi(t)-\xi(s),t-s) - G(x-y,t-s)\} \\ &\qquad\qquad \cdot V_n(y)u(y+\xi(s),s)dyds \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Lemma (Estimate of the heat kernel)

If $0 < \delta < 1$, then there exists a constant $C = C(\delta) > 0$ such that the following inequalities hold:

(i) $$|G(x-y+\xi(t)-\xi_0, t) - G(x-y, t)| \le Ct^{-N/2-1+\gamma}\{|x-y| + t^{\gamma}\} \exp\Big(-\frac{(1-\delta)|x-y|^2}{4t}\Big).$$

(ii) $$G(x-y+\xi(t)-\xi(s), t-s) \le \frac{1}{(4\pi(t-s))^{N/2}} \exp\Big(-(1-\delta)\frac{|x-y|^2}{4(t-s)}\Big).$$

(iii) $$|G(x-y+\xi(t)-\xi(s), t-s) - G(x-y, t-s)| \le C(t-s)^{-N/2-1+\gamma}\{|x-y| + (t-s)^{\gamma}\} \exp\Big(-\frac{(1-\delta)|x-y|^2}{4(t-s)}\Big).$$

Lemma (Estimate of the integrals)

There exists a constant $C = C(\delta) > 0$ and $R = R(\delta)$ independent of x, t, n such that the following inequalities hold for $|x| < R$:

(i) $|I_1| \leq C|x|^{2\gamma-1} \int_{\mathbb{R}^N} G(x-y, t/(1-2\delta))u_0(y+\xi(0))dy.$

(ii) $|I_2| \leq C|x|^{2\gamma-1}$
$$\cdot \int_0^t \int_{\mathbb{R}^N} G(x-y, (t-s)/(1-2\delta)) \cdot V_n(y)|w(y, t-s)|dyds.$$

(iii) $|I_3| \leq C|x|^{2\gamma-1}$
$$\cdot \int_0^t \int_{\mathbb{R}^N} G(x-y, (t-s)/(1-2\delta))V_n(y)u(y, t-s)dyds.$$

STEP 3: Gronwall-like argument

$$
\begin{aligned}
|w(x,t)| &= I_1 + I_2 + I_3 \\
&\le \int_0^t \int_{\mathbb{R}^N} G(x-y,(t-s)/(1-2\delta)) \frac{\lambda}{|y|^2} |w(y,s)| dyds \\
&\quad + C_1 |x|^{\gamma-1/2} u^+(x,t/(1-2\delta)) + C_2
\end{aligned}
$$

for all $x \in \mathbb{R}^N$. Let $W(x,t)$ denote the right-hand side of this inequality. Then we have

$$
\begin{aligned}
W_t \le \frac{1}{1-2\delta}\Delta W + \frac{\lambda}{|x|^2} W \\
+ C\Big\{ |x|^{\gamma-1/2} u_t^+(x,t/(1-2\delta)) - \Delta\big\{ |x|^{\gamma-1/2} u^+(x,t/(1-2\delta))\big\}\Big\}.
\end{aligned}
$$

This implies that W is a subsolution of

$$
\begin{aligned}
W_t = \frac{1}{1-2\delta}\Delta W + \frac{\lambda}{|x|^2} W \\
+ C\Big\{ |x|^{\gamma-1/2} u_t^+(x,t/(1-2\delta)) - \Delta\big\{ |x|^{\gamma-1/2} u^+(x,t/(1-2\delta))\big\}\Big\}.
\end{aligned}
$$

On the other hand,

$$W^+ := Ae^{At}|x|^{-\alpha_1}t^{-l}\varphi(\rho), \qquad \rho = (1-2\delta)^{1/2}t^{-1/2}|x|,$$

is a supersolution if $A > 0$ is sufficiently large. Since $\tilde{u}$ satisfies

$$\tilde{u} < u^+ + Ae^{At}|x|^{-\alpha_1}t^{-l}\varphi(\rho) \le Ct^{-l-1}|x|^{-\alpha_1}$$

for small $|x|$, the proof is (almost) complete. □

Theorem (Lower bound)

Assume

$$V(x,t) \geq \frac{\lambda}{|x-\xi(t)|^2}, \qquad 0 < |x-\xi(t)| < 1,$$

for some $\lambda > 0$. Then any positive solution satisfies

$$u(x,t) \geq C|x-\xi(t)|^{-\alpha_1(\lambda)}, \quad |x-\xi(t)| < 1.$$

Idea of the proof:

STEP 1: Consider the simplest case.
STEP 2: Compare with the moving singularity.
STEP 3: Gronwall-like argument.

$$u_t^- = \Delta u^- + \frac{\lambda}{|x|^2} u^-, \qquad x \neq 0.$$

Radial solution $u = v(r,t)$, $r = |x|$, satisfies

$$v_t = v_{rr} + \frac{N-1}{r} v_r + \frac{\lambda}{r^2} v, \quad r > 0.$$

Setting $w(r,t) := r^{\alpha_1} v(r,t)$, we have the radial heat equation

$$w_t = w_{rr} + \frac{d-1}{r} w_r, \qquad r > 0,$$

where $d = N - 2\alpha_1 > 2$.

Lemma (Positivity)

If $d \geq 2$, any nonnegative and nontrivial solution satisfies $w(r,t) > 0$ for $r \geq 0$ and $t > 0$.

Proof. Let $G^d(r,t)$ be the d-dimensional radial heat kernel defined by

$$G^d(q,r,t) := \int_{|y|=q} G(x-y,t)dy, \qquad r = |x|,$$

whichl is explicitly written as

$$G^d(q,r,t) = \frac{1}{4t(qr)^{d/2-1}} I_{d/2-1}(qr/2t)\exp(-\frac{q^2+r^2}{4t}),$$

where $I_{d/2-1}(z)$ is the modified Bessel function of the first kind of order $d/2-1$. Then

$$w^d(r,t) = \int_0^\infty G^d(qr,t)w_0(q)dq$$

satisfies the radial heat equation with

$$w^d_r(0,t) = 0, \qquad w^d(r,0) = w_0(r).$$

If $w_0(r) \geq 0$ and $w_0(r) \not\equiv 0$, then $w^d(r,t) > 0$ for all $r \geq 0$ and $t > 0$.

Lemma (Minimality)

$w^d(r,t)$ **is the minimal nonnegative solution.**

Proof. $w^d(r,t)$ **is a solution with the Neumann boundary condition at** $r = 0$**. We define a subsolution by**

$$w^-(r,t) = \max\{w(r,t) - \varepsilon r^{-d+2}, 0\}.$$

Here $w = r^{-d+2}$ **is a singular steady state. Hence for every** $\varepsilon > 0$**, we have** $w(r,t) > w^-(r,t)$ **for** $r > 0$ **and** $t > 0$**. Taking the limit as** $\varepsilon \downarrow 0$**, we obtain** $w(r,t) \geq w(r,t)$**. This proves the lemma.** $\square$

Summary for the existence

Heat equation with a dynamic Hardy term

$$u_t = \Delta u + V(x,t)u, \quad x \in \mathbb{R}^N \setminus \{\xi(t)\}.$$

Assumptions

- $V(x,t)$ is positive and continuous in $(x,t) \in \mathbb{R}^N \setminus \{\xi(t)\} \times [0,\infty)$, and is bounded for $|x| > 1$.
- $V(x,t)$ is singular at $\xi(t)$:

$$V(x,t) = \lambda(t)|x - \xi(t)|^{-2} + O(|x-\xi(t)|^{-2+\varepsilon}) \quad (x \to \xi(t)),$$

- $\xi = \xi(t)$ is γ-Hölder continuous with $\gamma > 1/2$.
- $\lambda(t)$ is a smooth positive function of $t \in [0,T]$.

If $\lambda(t) < \lambda_c$, the quadratic equation

$$\alpha^2 - (N-2)\alpha + \lambda(t) = 0$$

has two positive roots $0 < \alpha_1(t) < \alpha_2(t)$.

Theorem (Minimal solution)

(i) Assume

$$0 < V(x,t) \le \frac{\lambda}{|x-\xi(t)|^2}, \qquad 0 < |x-\xi(t)| < 1,$$

for some $0 < \lambda < \lambda_c$. If the initial value satisfies

$$u_0(x) \le C|x-\xi(0)|^{-k}, \qquad k < \alpha_2(\lambda) + 2 = N - \alpha_1(\lambda),$$

then there exists a minimal solution satisfying

$$u(x,t) \le C|x-\xi(t)|^{-\alpha_1(\lambda)}, \quad |x-\xi(t)| < 1.$$

(ii) Assume

$$V(x,t) \ge \frac{\lambda}{|x-\xi(t)|^2}, \qquad 0 < |x-\xi(t)| < 1,$$

for some $\lambda > 0$. Then any positive solution satisfies

$$u(x,t) \ge C|x-\xi(t)|^{-\alpha_1(\lambda)}, \quad |x-\xi(t)| < 1.$$

Corollary

Suppose that $\lambda(t) < \lambda_c$ for $t \in [0, T]$. If the initial value satisfies

$$u_0(x) \le C|x - \xi(0)|^{-k}, \qquad k < \alpha_2(\lambda) + 2 = N - \alpha_1(\lambda),$$

for some $k < \alpha_2(0) + 2$, then for any $\varepsilon > 0$, the minimal solution satisfies

$$c_1|x|^{-\alpha_1(t)+\varepsilon} \le u(x,t) \le c_2|x|^{-\alpha_1(t)-\varepsilon}, \qquad |x| < 1,$$

for every $t \in (0, T]$.

Corollary

Suppose that $\lambda(t) \equiv \lambda \in (0, \lambda_c)$ is constant. If initial value satisfies

$$u_0(x) \leq C|x - \xi(0)|^{-k}, \qquad k < \alpha_2(\lambda) + 2 = N - \alpha_1(\lambda),$$

for some $k < \alpha_2(0) + 2$, then the minimal solution satisfies

$$c_1|x|^{-\alpha_1} < u(x,t) < c_2|x|^{-\alpha_1}, \qquad |x| < 1,$$

for every $t \in (0, T]$.

Nonexistence

Theorem (Nonexistence)

If $\lambda(0) > \lambda_c$, then there are no positive solutions.

Proof. Consider the integral equation

$$u = T[u] := \int_{\mathbb{R}^N} G(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\frac{\lambda}{|y-\xi|^2}u(y,s)dyds.$$

Suppose $\lambda(0) > \lambda_c$. If $U > |x|^{-\alpha_1(0)}$ for $|x| < 1$, then

$$T[U] > (1+\delta)U(x,t) \quad |x| < 1.$$

□

[Other results]

- More precise asymptotics in the case $\lambda(t)$ depends on t.

$$u(x,t) \sim |x - \xi(t)|^{-\alpha_1(t)} (\log |x - \xi(t)|)^{\beta}$$

- Critical case $\lambda(t) = \lambda_c$.
- Existence of a solution with a stronger singularity

$$u \sim C|x - \xi(t)|^{-\alpha_2(t)}$$

- Uniqueness

$$u_1(x,0) = u_2(x,0),\ |u_1(x,t) - u_2(x,t)| = o(|x|^{-\alpha_1})$$
$$\Longrightarrow u_1 \equiv u_2.$$

- Classification

Part III: Nonlinear equations

[III-1: Nonlinear diffusion]

with Marek Fila, Jin Takahashi

Equation of porous medium type

$$u_t = \Delta u^m, \qquad x \in \mathbb{R}^N \setminus \{\xi(t)\}, \quad t > 0,$$

where $m > 0$ and $\xi \in C^1([0,\infty);\mathbb{R}^N)$.

Singular steady state

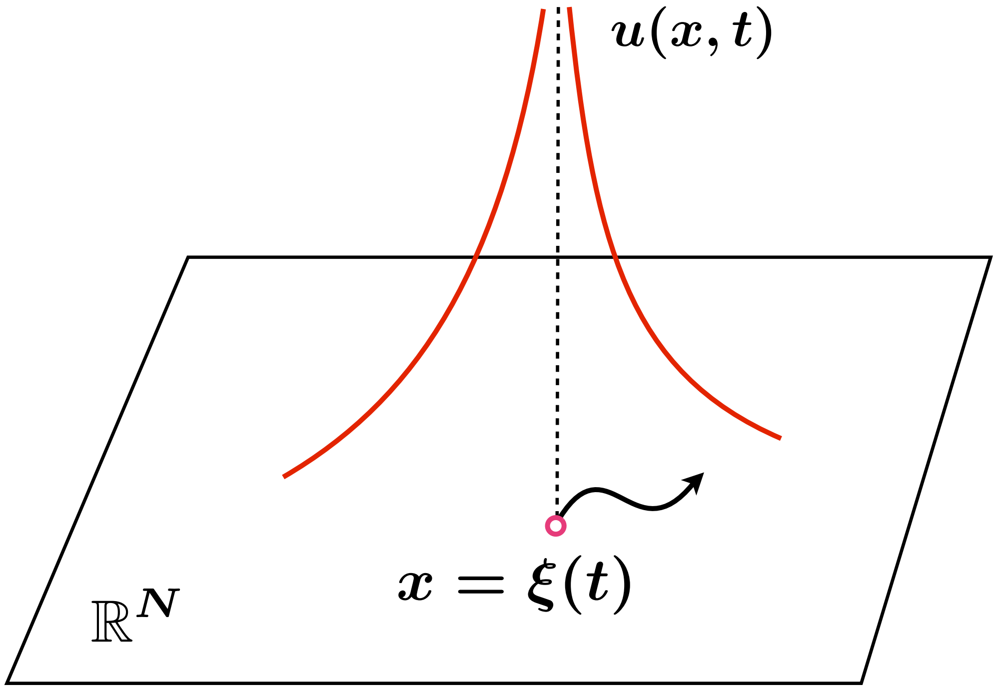
$$u = \varphi(x) := K|x|^{-\frac{N-2}{m}}, \qquad x \neq 0$$

where K is an arbitrary positive constant.

$$u_t = \Delta u^m = m\,\mathrm{div}(u^{m-1}\nabla u)$$

$m < 1 \implies$ **slow diffusion for large** u

$m > 1 \implies$ **fast diffusion for large** u



Known facts:

- **Vázquez-Winkler (2011):** $0 < m < \dfrac{N-2}{N}$

  Evolution of standing singularities of proper (minimal) solutions.

- **Lukkari (2010, 2012):** $m > \dfrac{N-2}{N-1}$

$$v_t - \Delta v^m = M(y,t),$$

where M **is a nonnegative Radon measure on** $\mathbb{R}^n \times \mathbb{R}$**.**

Consider

$$\boldsymbol{u_t = \Delta u^m, \qquad x \in \mathbb{R}^N \setminus \{\xi(t)\},}$$

where $\boldsymbol{\xi \in C^1}$ and the derivative $\boldsymbol{\xi'}$ is locally Hölder continuous.

Theorem (Existence)

Let $\boldsymbol{n \geq 3}$ and $\boldsymbol{m > m_* := (N-2)/(N-1)}$. Then for any positive function $\boldsymbol{k \in C^1}$, there exists a solution such that

$$\boldsymbol{v(y,t) = k(t)|x - \xi(t)|^{-\frac{N-2}{m}} + O(|x - \xi(t)|^{-\lambda})}$$

as $\boldsymbol{y \to \xi(t)}$ for each $\boldsymbol{t \geq 0}$, where $\boldsymbol{\lambda < (N-2)/m}$.

Remarks:

- $m = \dfrac{N-2}{N-1}$
  The critical case looks delicate. We have not found any obstacle for the existence, but our method cannot be modified easily.

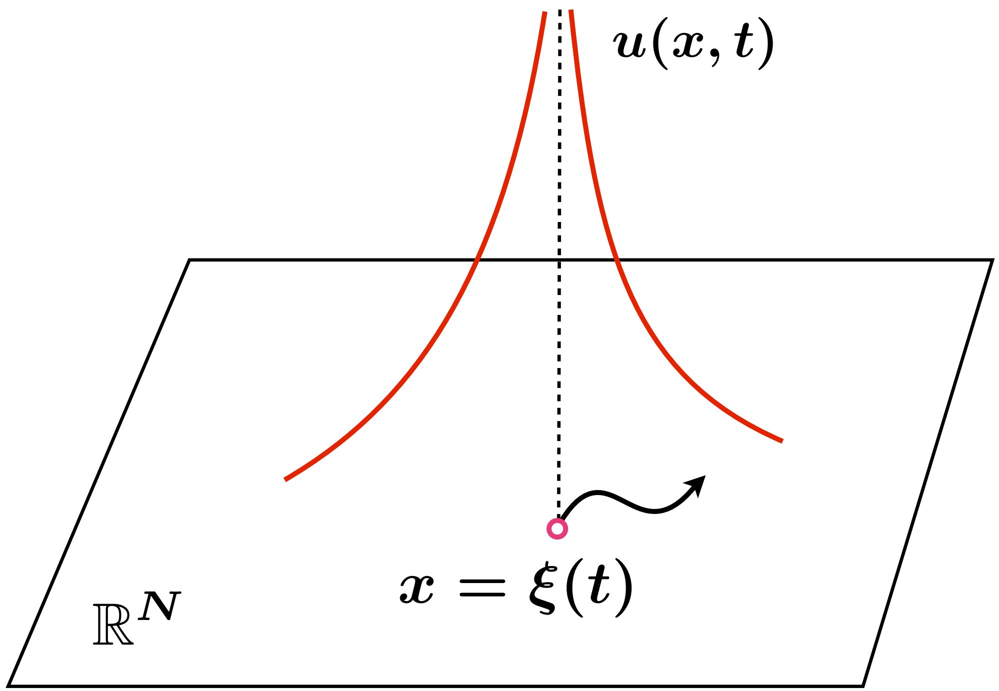
- $m < \dfrac{N-2}{N-1}$
  The result of Chasseigne (2003) on the "pressure equation" indicates that there is no solution with a moving singularity.

- $\dfrac{N-2}{N} < m < \dfrac{N-2}{N-1}$
  The problem is well-posed for a standing singularity, but there is no solution with a moving singularity.

- $m < \dfrac{N-2}{N}$
  Formal analysis suggests that the singularity is "half frozen". The singularity may NOT be asymptotically radially symmetric.

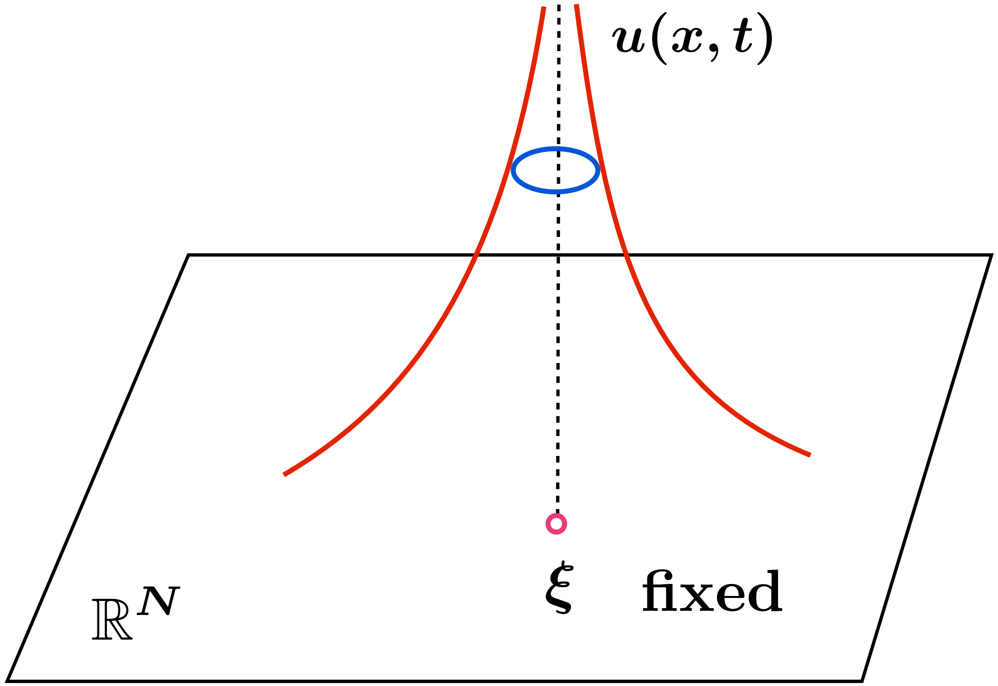
$$u_t = \Delta u^m = m\,\mathrm{div}(u^{m-1}\nabla u)$$

$$\frac{N-2}{N-1} < m < 1 \implies \quad \text{slow diffusion for large } u$$



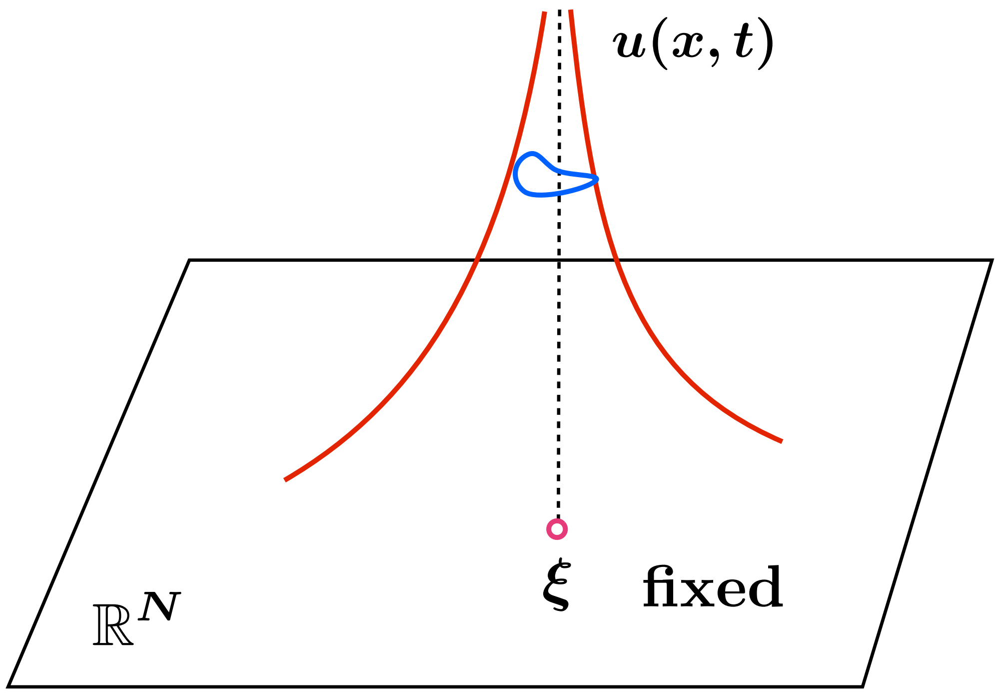
$$u_t = \Delta u^m = m \operatorname{div}(u^{m-1} \nabla u)$$

$$\frac{N-2}{N} < m < \frac{N-2}{N-1} \implies \text{very slow diffusion for large } u$$



$$u_t = \Delta u^m = m\,\mathrm{div}(u^{m-1}\nabla u)$$

$$m < \frac{N-2}{N} \implies \text{extremely slow diffusion for large } u$$



[III-2: Absorption equation]

with Jin Takahashi

Absorption equation

$$u_t = \Delta u - u^p$$

Stationary problem

$$\Delta u - u^p = 0, \quad x \neq \xi.$$

If $1 < p < \dfrac{N}{N-2}$, there is a radially symmetric singular solution

$$u = K|x - \xi|^{-\frac{2}{p-1}},$$

where

$$K = K(N, p) := \left\{ \left(\frac{2}{p-1} \right)^2 - \frac{2(N-2)}{p-1} \right\}^{\frac{1}{p-1}} > 0.$$

Veron (1981)

If $1 < p < \dfrac{N}{N-2}$, any isolated singularity is one of the following three types:

(i) Removable singularity.

(ii) $u(x) = c|x-\xi|^{2-N} + \cdots$, where c is an arbitrary constant.

(iii) $u(x) = K|x-\xi|^{-\frac{2}{p-1}} + \cdots$

Brezis–Veron (1980), Baras–Pierre (1984)

If $p \geq \dfrac{N}{N-2}$, then any isolated singularity is removable.

Removability of a moving singularity

Consider positive solutions of

$$u_t = \Delta u - u^p, \qquad x \neq \xi(t), \quad t \in (0, T).$$

Theorem (Removability)

Suppose that $\xi(t)$ is at least 1/2-Hölder continuous in $t \in [0, T]$.

(i) If $1 < p < \dfrac{N}{N-2}$ and

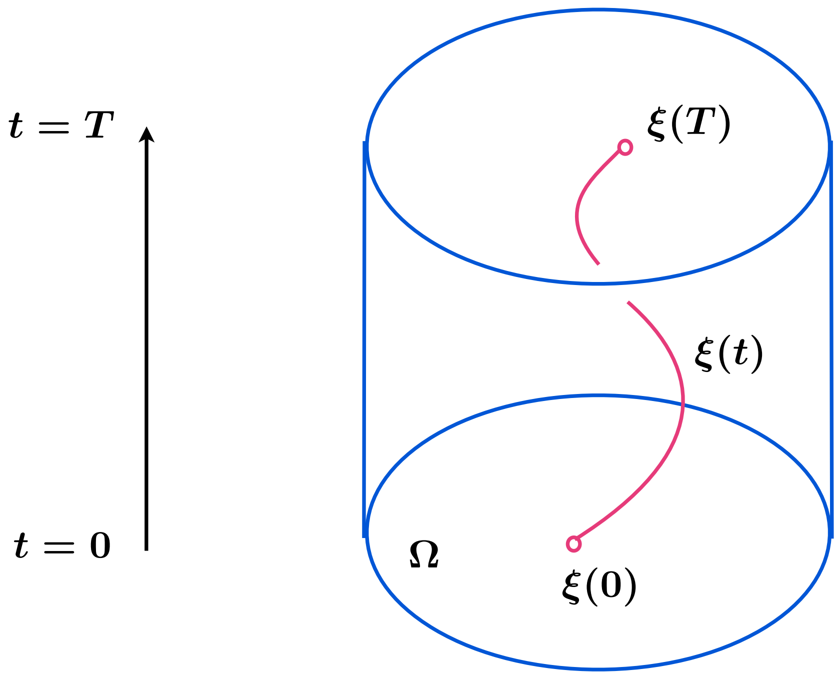
$$u(x,t) = o(|x - \xi(t)|^{-(N-2)}) \qquad (x \to \xi(t))$$

locally uniformly in $t \in (0, T)$, the singularity is removable.

(ii) If $p \geq \dfrac{N}{N-2}$, any singularity is removable.

Outline of the proof

STEP 1: By applying the method of Poláčik–Quittner–Souplet (2007), derive an a priori estimates which depend only on the parabolic distance from the boundary of a time-space domain.



Time-space domain

STEP 2: Use the estimate to show that u satisfies the absorption equation in $\Omega \times (0, T)$ in the distribution sense.

STEP 3: Apply the parabolic regularity theory by Brézis and Friedman (1983) to show $u \in L^\infty_{\text{loc}}(\Omega \times (0, T))$ and $u \in C^{2,1}(\Omega \times (0, T))$.

Classification of singularities

Formal asymptotic analysis suggests that non-removable singularities can be classified as follows:

- **Type F:** $u(x,t) = a(t)|x-\xi(t)|^{-(N-2)} + \cdots$.
  (Fundamental)

- **Type N:** $u(x,t) = K|x-\xi(t)|^{-\frac{2}{p-1}} + \cdots$.
  (Nonlinear)

- **Type A: Others**
  (Anomalous)

Singularities of Type F for

$$u_t = \Delta u - u^p \quad \text{on } \mathbb{R}^N \setminus \{\xi(t)\}, \quad t \in (0,T).$$

Theorem (Existence of Type F)

Let

$$1 < p < \frac{N}{N-2}.$$

Suppose that $\xi(t) \in C^1(0,T)$. Then for any positive function $a(t) \in C^1(0,T)$, there exists a singular solution if Type F:

$$u(x,t) = a(t)|x - \xi(t)|^{-(N-2)} + \cdots .$$

Outline of the proof

STEP 1: Let U be a solution of

$$U_t - \Delta U = a(t)\delta(x - \xi(t)) \qquad (x \in \mathbb{R}^N),$$

where $a(t) \in C^1(0, T)$. Then we have a singular solution such that

$$U(x, t) = C_N a(t)|x - \xi(t)|^{-(N-2)} + \cdots .$$

If $p < \dfrac{N}{N-2}$, then U is a nice approximate solution.

STEP 2: Construct suitable comparison functions by modifying the approximate solution U.

STEP 3: Construct a sequence of approximate solutions on annular domains, and show the convergence to the desired solution.

Singularities of Type N for

$$u_t = \Delta u - u^p \quad \text{on } \mathbb{R}^N \setminus \{\xi(t)\}, \quad t \in (0, T).$$

Theorem (Existence of Type N)

Let

$$1 < p < \frac{N}{N-2}.$$

Suppose that $\xi(t) \in C^1(0, T)$. Then there exists a singular solution of Type N:

$$u(x, t) = K|x - \xi(t)|^{-\frac{2}{p-1}} + \cdots .$$

Idea of the proof

Let U be a solution of

$$U_t - \Delta U = \delta(x - \xi(t)) \qquad (x \in \mathbb{R}^N).$$

Then we have a singular solution such that

$$U(x,t) = C_N |x - \xi(t)|^{-(N-2)} + \cdots .$$

The singular solution of (A) is well approximated by

$$u(x,t) \simeq K \left\{ \frac{U(x,t)}{C_N} \right\}^{\frac{2}{(p-1)(N-2)}} = K |x - \xi(t)|^{-\frac{2}{p-1}} + \cdots .$$

The remaining part of the proof is similar to the case of Type F.

Non-existence of Type A

Theorem (Non-existence of Type A)

Let

$$1 < p < \frac{N}{N-2}.$$

Suppose that $\xi(t)$ is 1/2-Hölder continuous in $t \in [0, T]$. If

$$u = \alpha(t)|x - \xi(t)|^{-\beta(t)} + \cdots$$

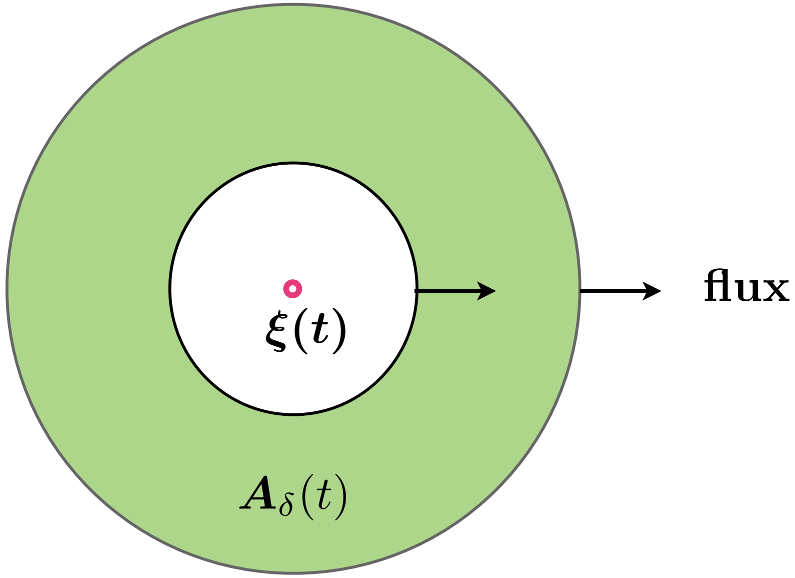
for some positive functions $\alpha(t) \in C(0, T)$ and $\beta(t) \in C^1(0, T)$. Then one of the following holds for $t \in (0, T)$:

(i) Type F: $\beta(t) \equiv N - 2$.

(ii) Type N: $\alpha(t) \equiv K$ and $\beta(t) \equiv \dfrac{2}{p-1}$.

Idea of the proof

Consider the balance of flux on an annular region.



Inward and outward flux.

The inward flux and the outward flux are balanced only if

$$u(x,t) = \alpha(t)|x-\xi(t)|^{-(N-2)} + \cdots$$

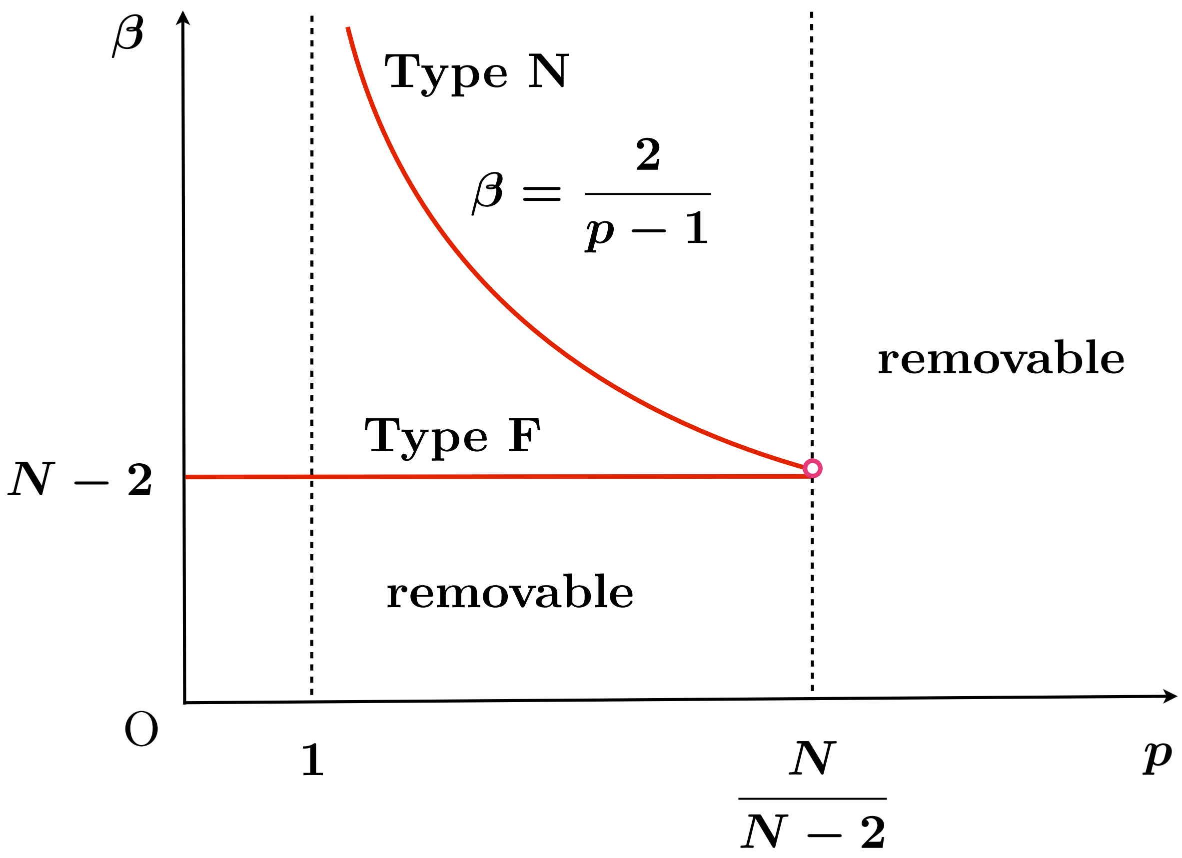
or

$$u(x,t) = K|x-\xi(t)|^{-\frac{2}{p-1}} + \cdots .$$

Summary for the absorption equation with a moving singularity

$$u_t = \Delta u - u^p \quad \text{on } D \setminus \{\xi(t)\}.$$

$$u(x,t) \sim |x - \xi(t)|^{-\beta}$$



[Part III-3: Fujita equation]

with Shota Sato

Fujita equation

$$u_t = \Delta u + |u|^{p-1} u.$$

Stationary problem (Lane-Emden equation)

$$\Delta u + u^p = 0, \quad u > 0 \quad \text{on } \mathbb{R}^N \setminus \{\xi\}.$$

- **There are radially symmetric singular solutions such that**

$$u = \begin{cases} C|x-\xi|^{-(N-2)} + \cdots & \text{for } p < \dfrac{N}{N-2}, \\ L|x-\xi|^{-\frac{2}{p-1}} & \text{for } p > \dfrac{N}{N-2}, \end{cases}$$

where $C > 0$ is an arbitrary constant and

$$L = L(N,p) := \left\{ -\left(\frac{2}{p-1}\right)^2 + \frac{2(N-2)}{p-1} \right\}^{\frac{1}{p-1}} > 0.$$

Gidas–Spruck (1981)

Let u be a stationary solution.

(i) If $1 < p \le \dfrac{N}{N-2}$, then any isolated singularity is removable.

(ii) Let $\dfrac{N}{N-2} < p < \dfrac{N+2}{N-2}$. If $u = o(|x-\xi|^{-\frac{2}{p-1}})$, then the singularity is removable.

Removability of a standing singularity $\xi(t) \equiv \xi_0$ for

Hirata–Ono (2014)

Let $1 < p < \dfrac{N}{N-2}$. The singularity is removable if and only if

$$u = o(|x-\xi_0|^{-(N-2)}) \qquad (x \to \xi_0).$$

Classification of singularities

$$u_t = \Delta u + u^p, \quad x \neq \xi(t).$$

Formal asymptotic analysis suggests that non-removable singularities can be classified as follows:

- **Type F:** $u(x,t) = a(t)|x - \xi(t)|^{-(N-2)} + \cdots .$
  (Fundamental)
- **Type N:** $u(x,t) = L|x - \xi(t)|^{-\frac{2}{p-1}} + \cdots .$
  (Nonlinear)
- **Type A: Others**
  (Anomalous)

Existence of a solution with a moving singularity

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N \setminus \{\xi(t)\},\ t \in (0, T).$$

Kan–Takahashi (2016) (Existence of Type F)

If $p < \dfrac{N}{N-2}$, then there exists a singular solution of Type F:

$$u(x,t) = a(t)|x - \xi(t)|^{-(N-2)} + \cdots .$$

Theorem (Existence of Type N)

If

$$\frac{N}{N-2} < p < p_c = \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}},$$

then there exists a singular solution of Type N:

$$u(x,t) = L|x - \xi(t)|^{-\frac{2}{p-1}} + a(t)|x - \xi(t)|^{-\lambda(N,p)} + \cdots .$$

Why $\frac{N}{N-2} < p < p_c$?

We formally expand the solution $u(x,t)$ in terms of $r = |x - \xi(t)|$ as follows:

$$u(x,t) = Lr^{-m} + a(t)r^{-\lambda} + \sum_{i=1}^{[m]} b_i(\omega,t)r^{-m+i} + v(y,t).$$

Substitute this expansion into the equation and equate each power of r to obtain a system of equations for $b_i(\omega,t)$. These equations are solvable and the remainder term $v(y,t)$ must satisfy

$$v_t = \Delta v + \xi_t \cdot \nabla v + \frac{pL^{p-1}}{|y|^2} v + o(|y|^{-2}).$$

This equation is well-posed if and only if

$$0 < pL^{p-1} < \frac{(N-2)^2}{4}.$$

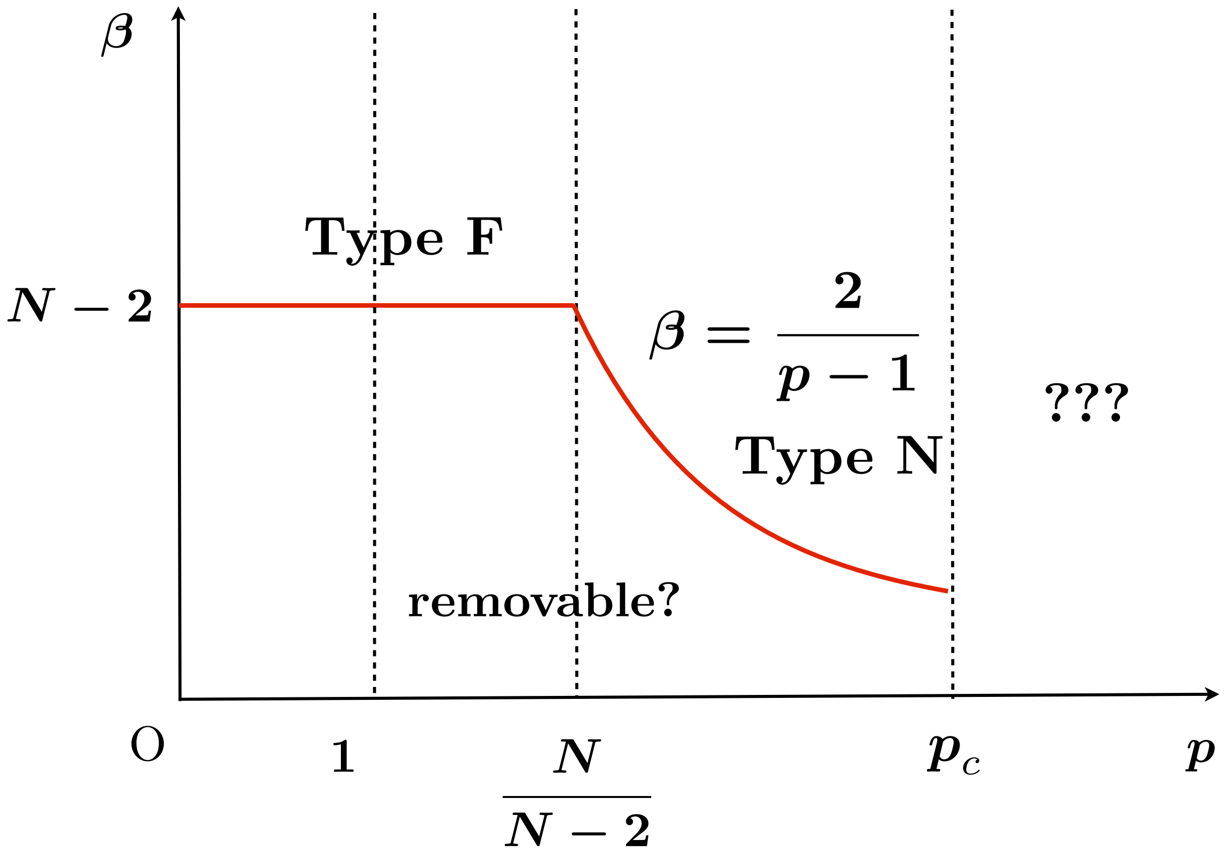
These inequalities hold if

$$N > 2 \quad \text{and} \quad \frac{N}{N-2} < p < p_c = \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}.$$

Summary for the Fujita equation

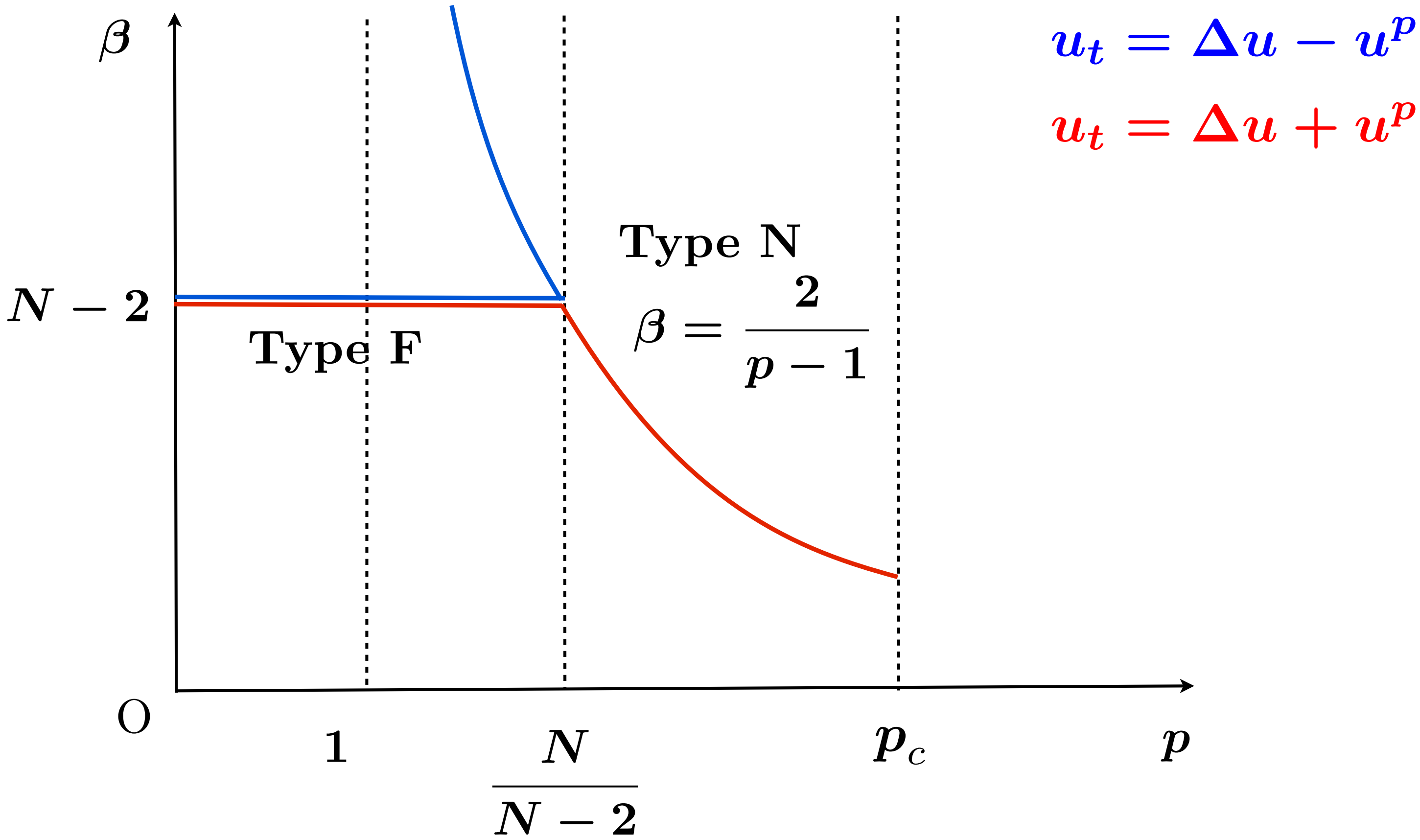
$$u_t = \Delta u + u^p, \quad x \neq \xi(t).$$

$$u(x,t) \sim |x - \xi(t)|^{-\beta}$$



Comparison of the absorption equation and the Fujita equation

$$u(x,t) \sim |x - \xi(t)|^{-\beta}$$



[Other results for the Fujita equation]

- **Time-global solution with a moving singularity.**

  ... Sato–Y (2010)

- **Sudden appearance of a moving singularity.** **... Sato (2011)**

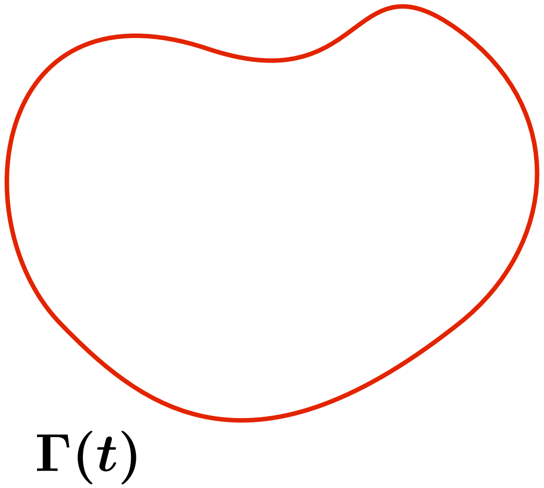
- **Emergence of an anomalous singularity.** **... Sato–Y (2012)**

- **Convergence to a singular steady state**

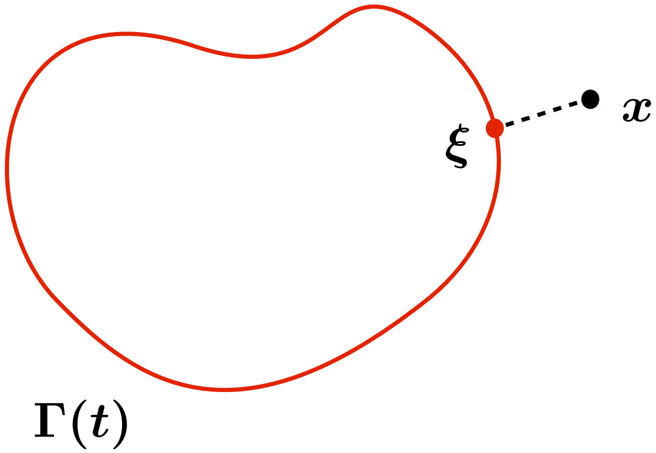
  ... Sato–Y (2012), Hoshino–Y (2016)

Part IV: Related topics

[Higher dimensional singularities]



$\Gamma(t)$ is a curve or a surface with codimension $\tilde{N} \geq 3$.



Htoo–Takahashi–Y (Higher dimensional singularity·

If $\tilde{N}/(\tilde{N}-2) < p < p_c(\tilde{N})$, then the Fujta equation has a solution of the form

$$u(x,t) = \tilde{L}|x-\xi|^{-\frac{2}{p-1}} + a(\xi,t)|x-\xi|^{-\lambda(\tilde{N},p)} + \cdots ,$$

where $\tilde{N}$ is the codimension, $\tilde{L} = \tilde{L}(\tilde{N},p)$, $\xi = \xi(x,t)$ is the nearest point on $\Gamma(t)$, and $a(\xi,t)$ is arbitrary.

The asymptotic profile depend on the distance from $\Gamma(t)$. For the proof, we need to consider the effect of the shape of $\Gamma(t)$.

Remarks.

- **Codimension 2: Logarithmic term appears in asymptotic profile.**

- **When $1 < p < N/(N-2)$, a quite general result was obtained by Kan-Takahashi (2016, 2017) for**

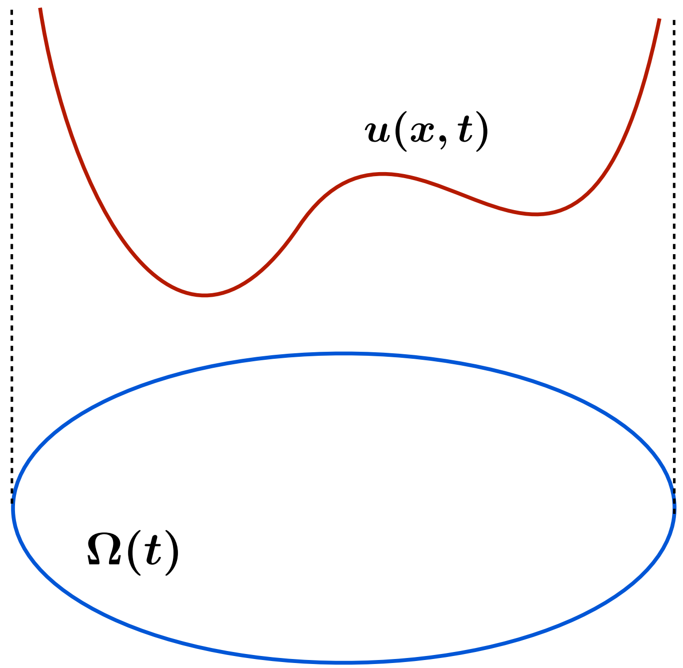
$$u_t - \Delta u = M(x,t),$$

where M is a nonnegative Radon measure on $\mathbb{R}^N \times \mathbb{R}$.

[Singulairity of codimension 1]

$$\begin{cases} u_t = \Delta u - f(u), & x \in \Omega(t),\ t > 0, \\ u \to +\infty, & x \to \partial\Omega(t),\ t > 0. \end{cases}$$

where $f \in C([0, \infty))$ is a nondecreasing nonnegative function and $\Omega(t)$ is a bounded domain in $\mathbb{R}^N$ depending on t.



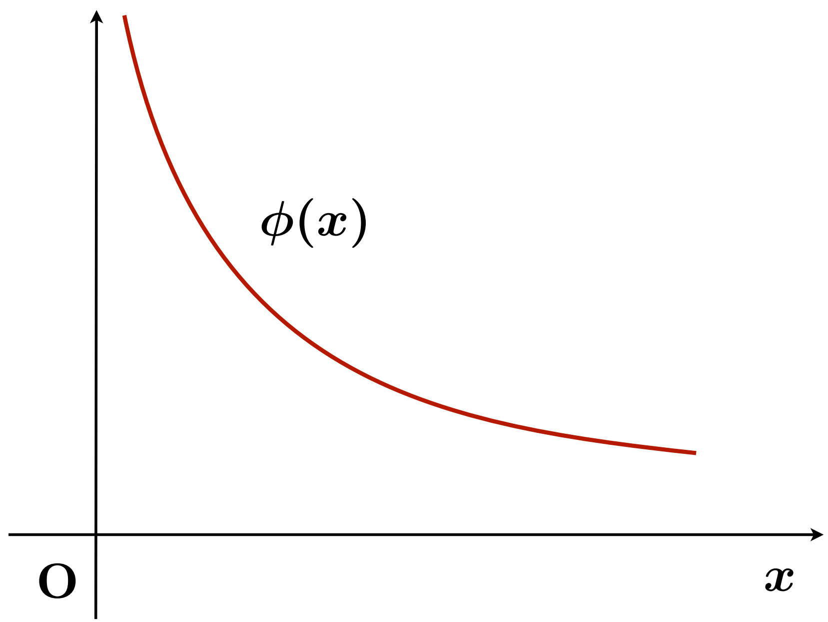
Large solution with a moving boundary.

For f, we assume $f(u) > 0$ and the Keller-Osserman condition

$$\int^{\infty} \frac{dt}{\sqrt{F(t)}} < \infty, \quad F(t) = \int_0^t f(s)\, ds.$$

The the one-dimensional problem has a solution:

$$\begin{cases} \phi''(x) - f(\phi) = 0, & x > 0, \\ \phi(x) \to \infty, & x \downarrow 0. \end{cases}$$



$$\begin{cases} u_t = \Delta u - f(u), & x \in \Omega(t),\ t > 0, \\ u \to +\infty, & x \to \partial\Omega(t),\ t > 0. \end{cases}$$

Bandle-Kan-Y (Large solution)

There exists a solution of the form

$$u(x,t) = \phi(d(x,t)) + o(d(x,t)) \quad \text{as } x \to \partial\Omega,$$

where

$$d(x,t) := \mathrm{dist}(x, \partial\Omega(t)) = \inf_{\xi \in \partial\Omega(t)} |x - \xi|, \quad x \in \Omega(t).$$

For the proof, we need to consider the effect of the motion and shape of $\partial\Omega(t)$, which appears in the second-order term. In fact, to construct suitable comparison functions, we use a solution of the equation

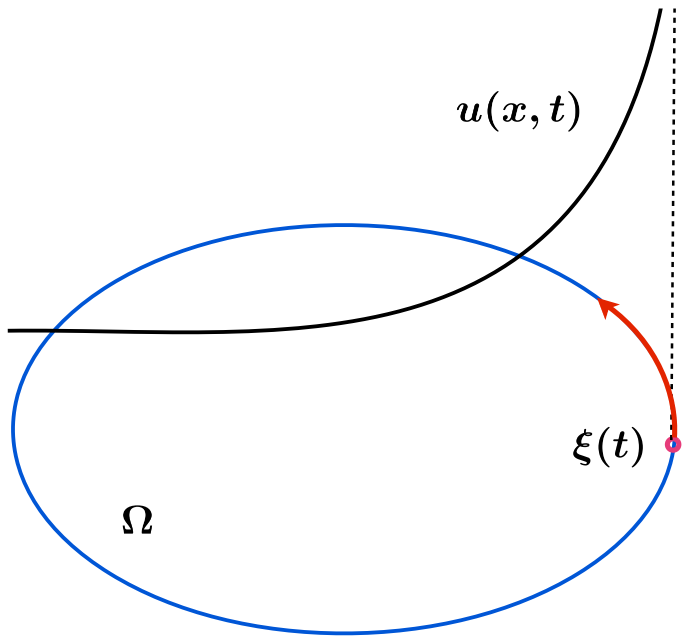
$$\phi'' - \mu\phi' - f(\phi) = 0,$$

where μ depends on the curvature and the normal velocity of $\partial\Omega(t)$.

[Point singularity on boundary]

$$\begin{cases} u_t = \Delta u + u^p, & x \in \Omega, \\ \dfrac{\partial}{\partial \nu} u = 0 & x \in \partial\Omega \setminus \{\xi(t)\}, \\ u(x,t) \to \infty, & x \to \xi(t), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain.



Moving singularity on the boundary.

Assumptions:

- $f(u) = u^p + O(u^q)$ as $u \to \infty$, where

$$p_{sg} < p < \begin{cases} p_* & \text{for } N \leq 5, \\ \dfrac{3N+3}{3N-5} & \text{for } N > 5, \end{cases}$$
$$0 \leq q < q_*(p) \ (< p).$$

- $\partial\Omega \in C^{1+\alpha}$ $(\alpha > 0)$.
- $\xi(t) \in C^1$.

Htoo-Y (Boundary singularity)

For any given C^1-function $a(t)$, there exists a solution of the form

$$u(x,t) = L|x - \xi(t)|^{-m} + a(t)|x - \xi(t)|^{-\lambda_2} + o(|x - \xi(t)|^{-\lambda_2})$$

as $x \to \xi(t)$, where $\lambda_2 = \lambda_2(N,p) < m$.

Remark:

- **Not only the motion of a singularity but also the curved boundary affect the asymptotic profile of the singularity.**
- **If $\partial\Omega \in C^{1+\alpha}$, then the boundary effect is minor.**

On-going projects and future plans

[Equations]

- **Other parameter regions**
- **Other equations (types, nonlinearities, nonlocal, anisotropic)**
- **Other boundary conditions**
- **Navier-Stokes**

  ... Karch-Zheng (2015), Kozono (?)

[Solutions]

- **Sign-changing solutions**
- **Sudden appearance and disappearance**
- **Collision and splitting**

  ... Nonuniqueness. Immediate regularization. Classification.
- **Traveling solutions, self-similar solutions, periodic solutions.**
- **Global existence and blow-up**

[Singularities]

- **More general singular set**
- **Anomalous singularity**
- **Dipole singularity, quadrupole singularity, hexapole singularity, octupole singularity, ... multipole singularity.**
- **Complicated motion of singularities**

  ... γ**-Hölder (**$\gamma < 1/2$**) continuity of** $\xi(t)$**.**

  Fractional Brownian motion

[Applications]

- **PDE theory**
- **Geometric flow**

  ... Harmonic flow, Ricci flow, Yamabe flow, Curvature flow
- **Stochastic process**
- **Modelling**

References

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