特異的領域変形と楕円型作用素の固有値の挙動 (の続き)

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細い弾性体の固有振動数 - 曲げ, 捩れ, 伸縮モードの漸近挙動

§0. Introduction - thin elastic body

Studies on the elastic properties of thin bodies are important from engineering point of view. Buildings, bridges, towers, aircrafts (and etc) are composed of several thin material. The mass of those objects are much smaller than they look like.

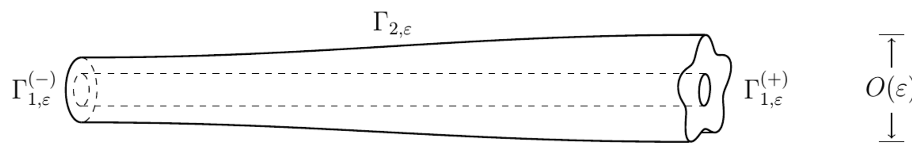


FIGURE 1. Sample of $\Omega(\varepsilon)$

§1. Spectral problem of elastic bodies

$\Omega \subset \mathbb{R}^3$: a bounded domain (homogeneous isotropic elastic body)

Variable $u = (u_1, u_2, u_3) : \Omega \longrightarrow \mathbb{R}^3$ displacement vector field (変形場)

σ_{ij} $(1 \leqq i, j \leqq 3)$: 物体内の応力テンソル

フックの法則 : $\sigma_{ij}(x) = \sum_{i,j,k,l=1}^{3} C_{ijkl}(x) e_{kl}(u)$
ここで

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (1 \leqq i, j \leqq 3) \quad \text{(Strain tensor, ねじれテンソル)}$$

弾性体が一様等方的な場合: $C_{ijkl} = \lambda_1 \delta(k,l)\delta(i,j) + 2\lambda_2 \delta(i,k)\delta(l,j)$

$$\sigma_{ij}(u) = \lambda_1 \operatorname{div}(u)\delta(i,j) + 2\,\lambda_2\, e_{ij}(u) \quad (1 \leqq i, j \leqq 3) \quad \text{(Stress tensor, 応力テンソル)}$$

Lamé constants (ラメ定数): $\lambda_1 > 0$ 体積弾性率, $\lambda_2 > 0$ 剛性率

[Operator of elasticity]

$$L[u]_i = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(u)}{\partial x_j} \quad (1 \leqq i \leqq 3), \ \text{(elasticity operator)}$$

It is rewritten as

$$L[u] = \lambda_2 \Delta u + (\lambda_1 + \lambda_2)\nabla(\operatorname{div} u)$$

The oscillation of the elastic body :

$$\varrho \, \frac{\partial^2 u}{\partial t^2} = L[u] \qquad \text{(Wave motion)}$$

with some boundary condition.

It has a time periodic solution $u(t,x) = e^{i\omega t}\Phi(x)$ which leads to

$$L[\Phi] + \varrho\,\omega^2 \Phi = 0$$

The operator L appears as a gradient operator of the energy functional $\mathcal{H}(u)$

$$\mathcal{H}(u) = \int_\Omega \left(\lambda_1 (\operatorname{div} u)^2 + 2\lambda_2 \sum_{1 \leqq i,j \leqq 3} e_{ij}(u)^2 \right) dx$$

$$\frac{1}{2}\frac{d}{d\epsilon}\mathcal{H}(u+\epsilon\,\varphi)_{|\epsilon=0} = (-L[u], \varphi)_{L^2(\Omega;\mathbb{R}^3)} \quad (\varphi \in C_0^\infty(\Omega;\mathbb{R}^3))$$

Time dependent problem

$$\mathcal{L}(u) = \int_0^T \int_\Omega (\frac{1}{2}\varrho|\frac{\partial u}{\partial t}|^2 - \frac{1}{2}\mathcal{H}(u))dxdt \quad \text{(Lagrangian)}$$

[Eigenvalue problem]

$$(E) \qquad \begin{cases} L[\Phi] + \mu\,\Phi = \mathbf{0} \quad \text{in} \quad \Omega, \\ \Phi = \mathbf{0} \quad \text{on} \quad \Gamma_1 \text{ (fixed)}, \quad \sigma(\Phi)\nu = \mathbf{0} \quad \text{on} \quad \Gamma_2 \text{ (No-traction)} \end{cases}$$

$\partial\Omega = \Gamma_1 \cup \Gamma_2,\ \Gamma_1 \cap \Gamma_2 = \emptyset$

[Max-Min principle for the eigenvalues]

$$\mu_1 = \inf\{R[\Phi] \mid \Phi \in W,\, \Phi \neq \mathbf{0})\}, \quad R[\Phi] := \mathcal{H}[\Phi]/\|\Phi\|^2_{L^2(\Omega;\mathbb{R}^3)} \quad \text{(Rayleigh quotient)}$$

$$W := \{\Phi \in H^1(\Omega;\mathbb{R}^3) \mid \Phi = \mathbf{0} \text{ on } \Gamma_1\}$$

$$\mathcal{W}_\ell := \{F \subset L^2(\Omega;\mathbb{R}^3) \mid F : \text{a linear subspace with, } \dim(F) \leqq \ell\}$$

The $k-$ eigenvalue is characterized as follows

Theorem (max-min principle).

$$\mu_k = \sup_{F \in \mathcal{W}_{k-1}} \left(\inf\{R[\Phi] \mid \Phi \in W,\, \Phi \perp F \,\text{in}\, L^2(\Omega;\mathbb{R}^3)\}\right)$$

The set $\{\mu_k\}_{k \geqq 1}$ becomes a system of the eigenvalues of (E). It is an non-decreasing unbounded sequence of real numbers. They are associated with a complete system of orthonormalized eigenfunctions $\{\Phi_k\}_{k=1}^{\infty}$.

Energy level of different oscillations of a thin rod

Question: What are the characteristic spectral properties of thin rods ?

Geometric Instinct tells that there could be phenomena of bending, stretching, torsion (at least) and it seems that bending modes easily occur.

$$\Omega(\epsilon) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 < \epsilon^2,\ 0 < x_3 < \ell\} \qquad \text{(thin domain, simple case)}$$

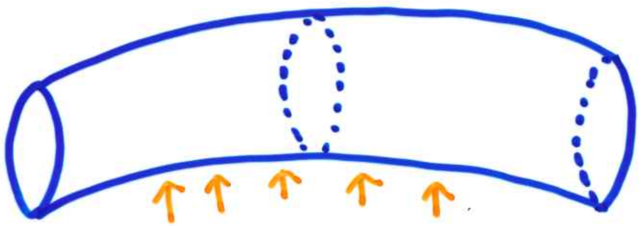
Rough characterization for several modes

(Stretching (伸縮) mode) $\quad \Phi(x) = (0,\ 0,\ \phi(x_3))$

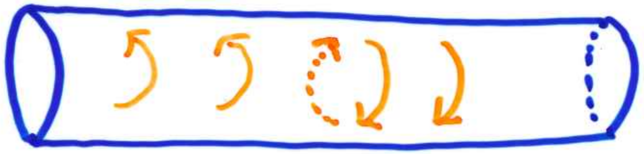
(Torsion (捩れ) mode) $\quad \Psi(x) = (-x_2\psi(x_3),\ x_1\psi(x_3),\ 0)$

(Bending (曲げ) mode) $\quad \Upsilon(x) = (0,\ \upsilon(x_3),\ -x_2\,\upsilon'(x_3))$

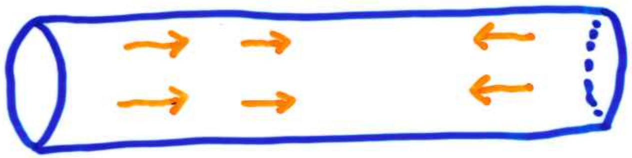
Here $\phi = \phi(\tau), \psi = \psi(\tau), \upsilon = \upsilon(\tau) \in C^1([0, \ell])$: arbitrary scalar functions with a certain boundary condition at $\tau = 0, \ell$



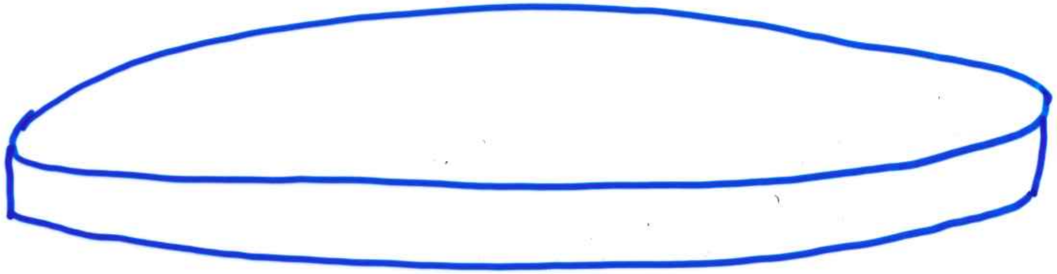
曲げ(Bending)モード



ねじれ(Torsion)モード



伸縮(stretching)モード



薄い板
Thin Slab

細い棒
Thin Rod

Simple calculation for evaluating "several kinds of deformation" and "energy cost".

$$\|\Phi\|^2_{L^2(\Omega(\epsilon):\mathbb{R}^3)} = \pi\epsilon^2 \int_0^\ell \phi(\tau)^2 d\tau, \quad \|\Psi\|^2_{L^2(\Omega(\epsilon):\mathbb{R}^3)} = \frac{\pi\epsilon^4}{2} \int_0^\ell \psi(\tau)^2 d\tau,$$

$$\|\Upsilon\|^2_{L^2(\Omega(\epsilon):\mathbb{R}^3)} = \pi\epsilon^2 \int_0^\ell \upsilon(\tau)^2 d\tau + \frac{\pi\epsilon^4}{4} \int_0^\ell (\upsilon'(\tau))^2 d\tau$$

$$\mathcal{H}(\Phi) = (\lambda_1+2\lambda)\epsilon^2 \int_0^\ell (\phi'(\tau))^2 d\tau, \quad \mathcal{H}(\Psi) = \pi\lambda_2\epsilon^4 \int_0^\ell (\psi'(\tau))^2 d\tau, \quad \mathcal{H}(\Upsilon) = \lambda_2 \frac{\epsilon^4\pi}{2} \int_0^\ell (\upsilon''(\tau))^2 d\tau$$

We see that

$$R(\Phi) = O(1), \quad R(\Psi) = O(1), \quad R(\Upsilon) = O(\epsilon^2)$$

Lower class of eigenvalues correspond to Bending modes.

§2. Small eigenvalues - Bending modes in thin rods

$\ell > 0$: a constant, $B \subset \mathbb{R}^2$: a connected bounded domain with C^3 boundary

$$S = B \times (0,\ell), \qquad s_1 = (\overline{B} \times \{0\}) \cup (\overline{B} \times \{\ell\}), \quad s_2 = \partial B \times (0,\ell).$$

$\partial S = s_1 \cup s_2$

$F : \mathbb{R}^3 \to \mathbb{R}^3$ (C^3-diffeomorphism) is defined as

$$F(z) = (F_1(z), F_2(z), z_3) \quad (z = (z_1, z_2, z_3) \in S)$$

$$F_i(0,0,z_3) = 0 \quad (0 \leqq z_3 \leqq \ell,\ i = 1,2), \quad \text{the Jacobian of } F(z) > 0 \quad (z \in S)$$

$$F_\epsilon(z) = (\epsilon F_1(z), \epsilon F_2(z), z_3) \quad (z = (z_1, z_2, z_3) \in S)$$

$$\Omega(\epsilon) = F_\epsilon(S), \quad \Gamma_1(\epsilon) = F_\epsilon(s_1), \quad \Gamma_2(\epsilon) = F_\epsilon(s_2).$$

Note $\partial\Omega(\epsilon) = \Gamma_1(\epsilon) \cup \Gamma_2(\epsilon)$.

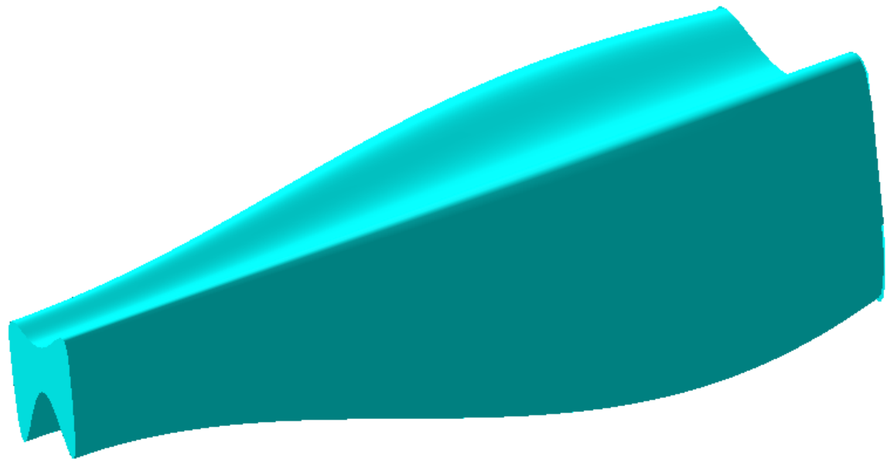


FIGURE 2. Sample of Ω

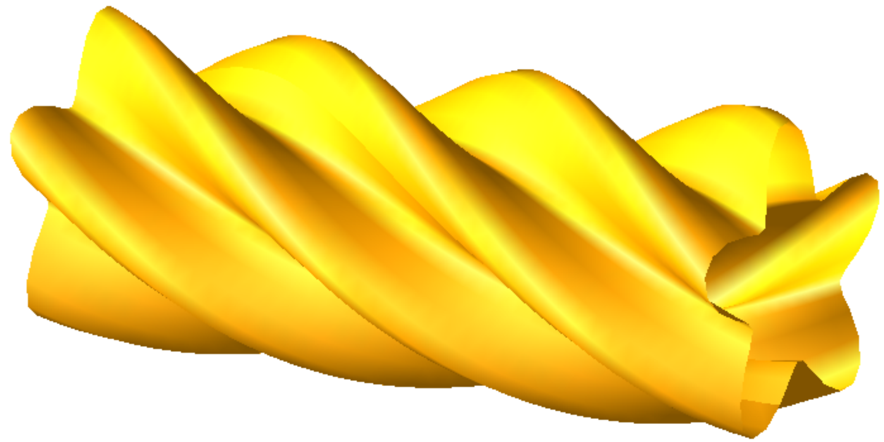


FIGURE 3. Sample of Ω

Coordinates

$$\Omega(\epsilon) \ni x = (x_1, x_2, x_3), \qquad \Omega \ni y = (y_1, y_2, y_3), \qquad S \ni z = (z_1, z_2, z_3)$$

$$(x_1, x_2, x_3) = (\epsilon y_1, \epsilon y_2, y_3), \quad (y_1, y_2, y_3) = (F_1(z), F_2(z), z_3), \quad (x_1, x_2, x_3) = (\epsilon F_1(z), \epsilon F_2(z), z_3)$$

Eigenvalue problem

$$(EP)_\epsilon \qquad \begin{cases} L[\Phi] + \mu\,\Phi = \mathbf{0} & \text{in} \quad \Omega(\epsilon) \\ \Phi = \mathbf{0} & \text{on} \quad \Gamma_1(\epsilon) \\ \sigma(\Phi)\,\boldsymbol{n} = \mathbf{0} & \text{on} \quad \Gamma_2(\epsilon) \end{cases}$$

Denote by $\{\mu_k(\varepsilon)\}_{k=1}^{\infty}$ the eigenvalues of $(EP)_\epsilon$ which are arranged in increasing order, counting multiplicities (cf. the books of Courant-Hilbert, Edmunds-Evans).

Preparation for the main results

Definition

$$\widehat{\Omega}(y_3) = \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1, y_2, y_3) \in \Omega\}$$

$$H(y_3) = \int_{\widehat{\Omega}(y_3)} 1\, dy_1 dy_2, \quad K_i(y_3) = \int_{\widehat{\Omega}(y_3)} y_i\, dy_1 dy_2, \quad A_{ij}(y_3) = \int_{\widehat{\Omega}(y_3)} y_i y_j dy_1 dy_2 (0 \leqq y_3 \leqq \ell)$$

$$\Pi_{ij} = \Pi_{ij}(y_3) = A_{ij}(y_3) - \frac{K_i(y_3) K_j(y_3)}{H(y_3)}, \quad Y = \frac{\lambda_2(3\lambda_1 + 2\lambda_2)}{\lambda_1 + \lambda_2} \quad \text{Young modulus.}$$

Remark. Denote

$$J(z) = \left(\frac{\partial F_i}{\partial z_j}\right)_{1 \leq i,j \leq 3} = \begin{pmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} & \frac{\partial F_1}{\partial z_3} \\ \frac{\partial F_2}{\partial z_1} & \frac{\partial F_2}{\partial z_2} & \frac{\partial F_2}{\partial z_3} \\ 0 & 0 & 1 \end{pmatrix}, \quad J_*(z) = \det(J(z))$$

then

$$H(z_3) = \int_B J_*(z', z_3) \mathrm{d}z', \qquad K_i(z_3) = \int_B F_i(z', z_3) J_*(z', z_3) \mathrm{d}z',$$

$$A_{ij}(z_3) = \int_B F_i(z', z_3) F_j(z', z_3) J_*(z', z_3) \mathrm{d}z' \quad (z_3 \in [0, l])\,.$$

Theorem (S. J., A.Rodríguez Mulet). For any $k \in \mathbb{N}$,

$$\limsup_{\epsilon \to 0} \mu_k(\epsilon)/\epsilon^2 < \infty \tag{1}$$

and we have the limit value

$$\lim_{\varepsilon \to 0} \frac{\mu_k(\varepsilon)}{\varepsilon^2} = \Lambda_k,$$

where Λ_k is the k-th eigenvalue of the 4th order ODE system (limit equation)

$$\begin{cases} Y \dfrac{d^2}{d\tau^2} \left(\begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) & -K_1(\tau) \\ A_{21}(\tau) & A_{22}(\tau) & -K_2(\tau) \end{pmatrix} \begin{pmatrix} \dfrac{d^2\eta_1}{d\tau^2} \\ \dfrac{d^2\eta_2}{d\tau^2} \\ \dfrac{d\eta_3}{d\tau} \end{pmatrix} \right) = \Lambda \, H(\tau) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} & (0 < \tau < l), \\ \dfrac{d}{d\tau} \left(H(\tau) \dfrac{d\eta_3}{d\tau} \right) = \dfrac{d}{d\tau} \left(K_1(\tau) \dfrac{d^2\eta_1}{d\tau^2} + K_2(\tau) \dfrac{d^2\eta_2}{d\tau^2} \right) & (0 < \tau < l), \\ \eta_3(0) = \eta_i(0) = \dfrac{d\eta_i}{d\tau}(0) = 0 & (i = 1, 2), \\ \eta_3(l) = \eta_i(l) = \dfrac{d\eta_i}{d\tau}(l) = 0 & (i = 1, 2). \end{cases}$$

Special case: Straight cylinder

$$B = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1^2 + z_2^2 < 1\}, \quad F = \text{Identity map}, \quad \Omega = B \times (0, \ell)$$

For this special case, it is easy to see

$$H = \pi, \quad K_i = 0, \quad A_{ij} = \frac{\pi}{4}\delta(i,j) \quad (1 \leqq i, j \leqq 2)$$

The limit equation split into

$$\begin{cases} \dfrac{Y}{4}\dfrac{d^4\eta_i}{d\tau^4} - \Lambda\eta_i = 0 \quad (0 < \tau < \ell,\ i = 1, 2), \\ \eta_3 \equiv 0, \\ \eta_i(0) = \eta_i(\ell) = 0, \quad \dfrac{d\eta_i}{d\tau}(0) = \dfrac{d\eta_i}{d\tau}(\ell) = 0 \quad (i = 1, 2). \end{cases}$$

Rough sketch of the proof of this special case

Transform the problem on $\Omega(\epsilon)$ into that on Ω through the change of the variable.
Scale change (of variables): $\Omega(\epsilon) \ni x \longleftrightarrow y \in \Omega$

$$y_1 = x_1/\epsilon, \quad y_2 = x_2/\epsilon, \quad y_3 = x_3$$
$$U_1(y) = (1/\epsilon)u_1(x), \quad U_2(y) = (1/\epsilon)u_2(x), \quad U_3(y) = (1/\epsilon^2)u_3(x)$$

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_j} + \frac{\partial U_j}{\partial y_i}\right)$$
$$e_{i3}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i}\right) = \frac{\epsilon}{2}\left(\frac{\partial U_i}{\partial y_3} + \frac{\partial U_3}{\partial y_i}\right) \quad (1 \leqq i, j \leqq 3)$$
$$e_{33}(u) = \frac{\partial u_3}{\partial x_3} = \epsilon^2 \frac{\partial U_3}{\partial y_3}$$

$$E_{ij}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_j} + \frac{\partial U_j}{\partial y_i}\right), \ E_{i3}(U) = \frac{1}{2}\left(\frac{\partial U_i}{\partial y_3} + \frac{\partial U_3}{\partial y_i}\right), \ E_{33}(U) = \frac{\partial U_3}{\partial y_3}$$

Inner product :

$$(U,V)_\epsilon = \int_\Omega (U_1V_1 + U_2V_2 + \epsilon^2 U_3V_3)dy$$

Admissible space :

$$W = \{U \in H^1(\Omega, \mathbb{R}^3) \mid U = 0 \quad \text{on} \quad s_1\}$$

Energy in Ω :

$$\widetilde{\mathcal{H}}_\epsilon(U) := \int_\Omega \{(\lambda_1(E_{11}(U) + E_{22}(U) + \epsilon^2 E_{33}(U))^2$$

$$+2\lambda_2(\sum_{i,j=1}^{2} E_{ij}(U)^2 + 2\epsilon^2 \sum_{i=1}^{2} E_{i3}(U)^2 + \epsilon^4 E_{33}(U)^2)\}dy$$

Rayleigh quotient :

$$\widetilde{R}_\epsilon(U) = \widetilde{\mathcal{H}}_\epsilon(U) / \int_\Omega (\epsilon^2 U_1^2 + \epsilon^2 U_2^2 + \epsilon^4 U_3^2)\, dy$$

Max-Min principle (transformed)

$$\mu_k(\epsilon) = \sup_{Z \subset L^2(\Omega,\mathbb{R}^3),\dim Z \leqq k-1} \left(\inf\{\widetilde{R}_\epsilon(u) \mid u \in W,\ u \perp_\epsilon Z\} \right)$$

Proposition 1.

$$\mu_k(\epsilon) = O(\epsilon^2) \quad (\forall k \in \mathbb{N})$$

These small eigenvalues correspond to Bending modes. The max-min principle for eigenvalues applies for the proof.

(Sketch of the proof) Take test functions

$$\Upsilon^{(s)}(y) = (\eta_1^{(s)}(y_3), \eta_2^{(s)}(y_3), \eta_3^{(s)}(y_3) - y_1 \frac{d\eta_1^{(s)}}{dy_3} - y_2 \frac{d\eta_2^{(s)}}{dy_3})$$

such that $\eta_1^{(s)}, \eta_2^{(s)} \in H^2((0,\ell))$, $\eta_3^{(s)} \in H^1((0,\ell))$ $(s \geqq 1)$ are linearly independent and satisfy

$$\eta_i^{(s)}(0) = \eta_i^{(s)}(l) = 0 \quad (i = 1,2,3), \quad \frac{d\eta_i^{(s)}}{dy_3}(0) = \frac{d\eta_i^{(s)}}{dy_3}(l) = 0 \quad (i = 1,2).$$

and see $E_{ij}(\Upsilon^{(s)}) = 0$, $E_{i3}(\Upsilon^{(s)}) = 0$ $(i, j = 1, 2)$.

$$\widetilde{Z} = L.H. \left[\Upsilon^{(1)}, \Upsilon^{(2)}, \ldots, \Upsilon^{(k)} \right] \subset W.$$

$$\mu_k(\epsilon) = \sup_{\dim Z \leqq k-1, Z \subset L^2(\Omega, \mathbb{R}^3)} \inf\{ \widetilde{R}_\epsilon(\psi) \mid \psi \in W, \ \psi \perp_\epsilon Z \}$$

To use the above formula, take any subspace $Z \subset L^2(\Omega, \mathbb{R}^3)$ with $\dim Z \leqq k - 1$. Since $\dim Z < \dim \widetilde{Z}$, there exist $\Psi \in \widetilde{Z} \cap Z^{\perp_\epsilon}$, a vector $(c_1, \ldots, c_k) = (c_1(\epsilon), \ldots, c_k(\epsilon)) \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ such that

$$\Psi = \sum_{s=1}^{k} c_s(\epsilon) \Upsilon^{(s)}.$$

Substitute Ψ into $\widetilde{\mathcal{R}}_\epsilon(\Psi)$ and estimate the value with noticing that the estimate does not depend on a choice $(c_1, \ldots, c_k)$, we get $\widetilde{\mathcal{R}}_\epsilon(\Psi) \leqq C\epsilon^2$. The constant C can be taken independetly of Z. Accordingly we conclude

$$\mu_k(\epsilon) \leqq \exists C\epsilon^2 \quad (0 < \epsilon).$$

□

The proof consists of 2 parts.

(I) Convergence of the eigenfunction $\Phi_\epsilon^{(k)}$ and characterization $\longrightarrow$
Estimates of $\mu_k(\epsilon)/\epsilon^2$ from below.

(II) Construction of approximate eigenfunction $\longrightarrow$
Estimates of $\mu_k(\epsilon)/\epsilon^2$ from above.

(I) Convergence of eigenfunctions

Weak formulation of the eigenvalue equation:

$$\int_\Omega \Big\{ \lambda_1 \left(E_{11}(U) + E_{22}(U) + \epsilon^2 E_{33}(U)\right) \left(E_{11}(V) + E_{22}(V) + \epsilon^2 E_{33}(V)\right)$$
$$+ 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(U)E_{ij}(V) + 2\epsilon^2 \sum_{i=1}^{2} E_{i3}(U)E_{i3}(V) + \epsilon^4 E_{33}(U)E_{33}(V) \right) \Big\} dy$$
$$= \mu \int_\Omega \left(\epsilon^2 U_1 V_1 + \epsilon^2 U_2 V_2 + \epsilon^4 U_3 V_3\right) dy.$$

for any $V \in W$.

$k-$th eigenfunction:

$$\Phi_\epsilon^{(k)}(x) = (\Phi_{1,\epsilon}^{(k)}(x), \Phi_{2,\epsilon}^{(k)}(x), \Phi_{3,\epsilon}^{(k)}(x))$$

Eigenfunction corresponding to the eigenvalue $\mu_k(x)$ of (EP) in $\Omega(\epsilon)$. This is transformed to $U_\epsilon^{(k)}(y)$ through the above change of variables.

$$\int_\Omega \Bigg\{ \lambda_1 \left(E_{11}(U_\epsilon^{(k)}) + E_{22}(U_\epsilon^{(k)}) + \epsilon^2 E_{33}(U_\epsilon^{(k)}) \right)^2 + 2\lambda_2 \left(\sum_{i,j=1}^{2} E_{ij}(U_\epsilon^{(k)})^2 + 2\epsilon^2 \sum_{i=1}^{2} E_{i3}(U_\epsilon^{(k)})^2 + \epsilon^4 E_{33}(U_\epsilon^{(k)})^2 \right) \Bigg\} dy$$
$$= \mu_k(\epsilon) \int_\Omega \left(\epsilon^2 (U_{1,\epsilon}^{(k)})^2 + \epsilon^2 (U_{2,\epsilon}^{(k)})^2 + \epsilon^4 (U_{3,\epsilon}^{(k)})^2 \right) dy.$$

We can assume that

$$\|U_{1,\epsilon}^{(k)}\|_{L^2(\Omega)}^2 + \|U_{2,\epsilon}^{(k)}\|_{L^2(\Omega)}^2 + \|U_{3,\epsilon}^{(k)}\|_{L^2(\Omega)}^2 = 1 \quad \text{(Normalization)}$$

$$
\begin{aligned}
2\lambda_2\Big(\sum_{i,j=1}^{2} \|E_{ij}(U_\epsilon^{(k)})\|^2_{L^2(\Omega)} + 2\epsilon^2 \sum_{i=1,2} \|E_{i3}(U_\epsilon^{(k)})\|^2_{L^2(\Omega)} + \epsilon^4 \|E_{33}(U_\epsilon^{(k)})\|^2_{L^2(\Omega)}\Big) \\
\leqq \mathcal{H}(U_\epsilon^{(k)}) \int_\Omega (\epsilon^2 U_{1,\epsilon}^2 + \epsilon^2 U_{2,\epsilon}^2 + \epsilon^4 U_{3,\epsilon}^2)\, dy \\
= \mu_k(\epsilon) \int_\Omega (\epsilon^2 U_{1,\epsilon}^2 + \epsilon^2 U_{2,\epsilon}^2 + \epsilon^4 U_{3,\epsilon}^2)\, dy = O(\epsilon^4)
\end{aligned}
$$

By the Korn inequality:

$$
\|U\|^2_{H^1(\Omega,\mathbb{R}^3)} \leqq c(\Omega) \sum_{i,j=1}^{3} \|E_{ij}(U)\|^2_{L^2(\Omega)} \quad (U \in W),
$$

we obtain $\|U_\epsilon^{(k)}\|^2_{H^1(\Omega,\mathbb{R}^3)} \leqq \exists c(k)$ $(k \geqq 1)$ and $\exists\{\epsilon(p)\}_{p=1}^{\infty}$ with $\epsilon(p) \downarrow 0$ as $p \to \infty$ such that

$$
\lim_{p\to\infty} U_{\epsilon(p)}^{(k)} = \exists U^{(k)} \quad \text{weakly in} \quad H^1(\Omega, \mathbb{R}^3)
$$

$$
\lim_{p\to\infty} U_{\epsilon(p)}^{(k)} = U^{(k)} \quad \text{strongly in} \quad L^2(\Omega, \mathbb{R}^3)
$$

Moreover we have

$$E_{ij}(U_\epsilon^{(k)}) = O(\epsilon^2), \quad E_{i3}(U_\epsilon^{(k)}) = O(\epsilon), \quad E_{33}(U_\epsilon^{(k)}) = O(1) \quad (i,j=1,2).$$

Put, for $i,j=1,2$,

$$\kappa_{ij}^\epsilon = \frac{1}{\epsilon^2}E_{ij}(U_\epsilon^{(k)}), \quad \kappa_{i3}^\epsilon = \frac{1}{\epsilon}E_{i3}(U_\epsilon^{(k)}), \quad \kappa_{33}^\epsilon = E_{33}(U_\epsilon^{(k)})$$

and $\kappa_{ij}(\epsilon)$ is bounded in $L^2(\Omega)$ for each i,j. Taking subsequence of $\{\epsilon(p)\}_{p=1}^\infty$ (but we use the same notation)

$$\lim_{p\to\infty} \kappa_{ij}^{\epsilon(p)} = \exists \kappa_{ij} \text{ weakly in } \ L^2(\Omega)) \quad (1 \leqq i,j \leqq 3), \quad \lim_{p\to\infty} \frac{\mu_k(\epsilon(p))}{\epsilon(p)^2} = \exists \widetilde{\Lambda}_k.$$

Using this notation, we have

$$\int_\Omega \Bigg\{ \lambda_1(\kappa_{11}^{\epsilon(p)} + \kappa_{22}^{\epsilon(p)} + \kappa_{33}^{\epsilon(p)}) \left(E_{11}(V) + E_{22}(V) + \epsilon(p)^2 E_{33}(V)\right)$$

$$+ 2\lambda_2 \left(\sum_{i,j=1}^{2} \kappa_{ij}^{\epsilon(p)} E_{ij}(V) + 2\epsilon(p) \sum_{i=1}^{2} \kappa_{i3}^{\epsilon(p)} E_{i3}(V) + \epsilon(p)^2 \kappa_{33}^{\epsilon(p)} E_{33}(V) \right) \Bigg\} dy \quad V \in W$$

$$= \mu_k(\epsilon(p)) \int_\Omega \left(U_{1,\epsilon(p)}^{(k)} V_1 + U_{2,\epsilon(p)}^{(k)} V_2 + \epsilon(p)^2 U_{3,\epsilon(p)}^{(k)} V_3 \right) dy$$

Let $p \to \infty$, get

$$\int_\Omega \left(\lambda_1(\kappa_{11} + \kappa_{22} + \kappa_{33}) \left(E_{11}(V) + E_{22}(V)\right) + 2\lambda_2 \sum_{i,j=1}^{2} \kappa_{ij} E_{ij}(V) \right) dy = 0.$$

Next $V_2 = 0$, we see that $E_{22}(V) = 0$, and since $\kappa_{12} = \kappa_{21}$, it follows

$$\int_\Omega \left\{ \lambda_1 \sum_{p=1}^{3} \kappa_{pp} \frac{\partial V_1}{\partial y_1} + 2\lambda_2 \left(\kappa_{11} \frac{\partial V_1}{\partial y_1} + \kappa_{12} \frac{\partial V_1}{\partial y_2} \right) \right\} dy = 0$$

$$\int_\Omega \left\{ \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) \frac{\partial V_1}{\partial y_1} + 2\lambda_2 \kappa_{12} \frac{\partial V_1}{\partial y_2} \right\} dy = 0.$$

By integration by parts we obtain

$$-\int_\Omega \left\{ \frac{\partial}{\partial y_1} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) V_1 + \frac{\partial}{\partial y_2} \left(2\lambda_2 \kappa_{12} \right) V_1 \right\} dy = 0$$

$$-\int_\Omega \left\{ \frac{\partial}{\partial y_1} \left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{11} \right) + \frac{\partial}{\partial y_2} \left(2\lambda_2 \kappa_{12} \right) \right\} V_1 dy = 0.$$

Due to the arbitrariness of V_1 we have

$$\frac{\partial}{\partial y_1}\left(\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2\kappa_{11}\right)+\frac{\partial}{\partial y_2}\left(2\lambda_2\kappa_{12}\right)=0$$

in the distribution sense. Similarly, letting $V_1=0$ we also deduce that

$$\int_\Omega\left\{\left(2\lambda_2\kappa_{12}\right)\frac{\partial V_2}{\partial y_1}+\left(\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2\kappa_{22}\right)\frac{\partial V_2}{\partial y_2}\right\}dy=0,$$

$$\frac{\partial}{\partial y_1}\left(2\lambda_2\kappa_{12}\right)+\frac{\partial}{\partial y_2}\left(\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2\kappa_{22}\right)=0.$$

We write

$$\alpha_1=\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2\kappa_{11},\qquad \alpha_2=2\lambda_2\kappa_{12},$$

$$\beta_1=2\lambda_2\kappa_{12},\beta_2=\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2\kappa_{22}.$$

$$\int_\Omega \left(\alpha_1 \frac{\partial V_1}{\partial y_1} + \alpha_2 \frac{\partial V_1}{\partial y_2}\right) dy = 0, \quad \int_\Omega \left(\beta_1 \frac{\partial V_2}{\partial y_1} + \beta_2 \frac{\partial V_2}{\partial y_2}\right) dy = 0,$$
$$\frac{\partial \alpha_1}{\partial y_1} = -\frac{\partial \alpha_2}{\partial y_2}, \quad \frac{\partial \beta_1}{\partial y_1} = -\frac{\partial \beta_2}{\partial y_2}.$$

For every $\phi \in H^1(\Omega)$ with $\phi = 0$ on s_1, we have

$$\int_\Omega \left(\alpha_1 \frac{\partial \phi}{\partial y_1} + \alpha_2 \frac{\partial \phi}{\partial y_2}\right) dy = 0, \quad \int_\Omega \left(\beta_1 \frac{\partial \phi}{\partial y_1} + \beta_2 \frac{\partial \phi}{\partial y_2}\right) dy = 0.$$

Fact : There exist functions $h_1, h_2 \in L^2(\Omega)$ such that $\frac{\partial h_p}{\partial y_j} \in L^2(\Omega)$ for $1 \leq j, p \leq 2$ and

$$\frac{\partial h_1}{\partial y_1} = -\alpha_2, \; \frac{\partial h_1}{\partial y_2} = \alpha_1, \; \frac{\partial h_2}{\partial y_1} = -\beta_2, \; \frac{\partial h_2}{\partial y_2} = \beta_1.$$

Moreover, h_1, h_2 take values on the boundary and $h_p \mid s_2 \in L^2(s_2)$ for $p = 1, 2$.

$$\frac{\partial h_1}{\partial y_1} + \frac{\partial h_2}{\partial y_2} = \beta_1 - \alpha_2 = 0,$$
$$\frac{\partial h_1}{\partial y_2} - \frac{\partial h_2}{\partial y_1} = \alpha_1 + \beta_2 = 2\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2(\kappa_{11} + \kappa_{22}).$$

Let us write

$$Q = \frac{\partial h_1}{\partial y_2} - \frac{\partial h_2}{\partial y_1}.$$

$$Q = 2\left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + \lambda_2(\kappa_{11} + \kappa_{22})\right) = 2\left((\lambda_1 + \lambda_2) \sum_{p=1}^{3} \kappa_{pp} - \lambda_2 \kappa_{33}\right)$$

$$\lambda_1 Q = 2\left(\lambda_1(\lambda_1 + \lambda_2) \sum_{p=1}^{3} \kappa_{pp} - \lambda_1 \lambda_2 \kappa_{33}\right)$$

$$\lambda_1 Q + 2\lambda_2(3\lambda_1 + 2\lambda_2)\kappa_{33} = 2(\lambda_1 + \lambda_2)\left(\lambda_1 \sum_{p=1}^{3} \kappa_{pp} + 2\lambda_2 \kappa_{33}\right).$$

Eventually, we obtain

$$\frac{\lambda_1}{2(\lambda_1+\lambda_2)}Q+\frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}\kappa_{33}=\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2\kappa_{33}.$$

Try one more test function $V=(V_1,V_2,V_3)\in W$ where

$$V_1(y)=\rho_1(y_3),\quad V_2(y)=\rho_2(y_3),\quad v_3(y)=\rho_3(y_3)-y_1\frac{d\rho_1}{dy_3}-y_2\frac{d\rho_2}{dy_3}$$

with $\rho_1,\rho_2\in H^2((0,\ell))$, $\rho_3\in H^1((0,\ell))$ and

$$\rho_i(0)=\rho_i(l)=0\quad (i=1,2,3),\quad \frac{d\rho_i}{dy_3}(0)=\frac{d\rho_i}{dy_3}(l)=0\quad (i=1,2).$$

It is easy to see $E_{ij}(V)=0$, $E_{i3}(V)=0$ for $1\leqq i,j\leqq 2$.

$$\int_\Omega\left(\lambda_1\sum_{p=1}^{3}\kappa_{pp}+2\lambda_2\kappa_{33}\right)E_{33}(V)dy=\widetilde{\Lambda}_k\int_\Omega\left(U_1^{(k)}\rho_1+U_2^{(k)}\rho_2\right)dy.$$

$$\int_\Omega\left(\frac{\lambda_1}{2(\lambda_1+\lambda_2)}Q+\frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}\kappa_{33}\right)E_{33}(V)\mathrm{d}y=\widetilde{\Lambda}_k\int_\Omega\left(U_1^{(k)}\rho_1+U_2^{(k)}\rho_2\right)dy.$$

Note

$$E_{33}(V)=\frac{\partial V_3}{\partial y_3}=\frac{d\rho_3}{dy_3}-y_1\frac{d^2\rho_1}{dy_3^2}-y_2\frac{d^2\rho_2}{y_3^2}.$$

and

$$\int_{\widehat{\Omega}(y_3)} Q\mathrm{d}y'=0, \qquad \int_{\widehat{\Omega}(y_3)} Qy_i\mathrm{d}y'=0 \quad (i=1,2).$$

Using these facts, we see that

$$\int_\Omega QE_{33}(V)dy=0.$$

$$\int_\Omega \frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}\kappa_{33}E_{33}(V)\mathrm{d}y=\widetilde{\Lambda}_k\int_{F(S)}\left(U_1^{(k)}\rho_1+U_2^{(k)}\rho_2\right)dy.$$

Young modulus $Y=\frac{\lambda_2(3\lambda_1+2\lambda_2)}{\lambda_1+\lambda_2}$ appear here.

Here we compute κ_{33}. From $E_{ij}(U)=0$, $E_{i3}(U)=0$ $(1 \leqq i, j \leqq 2)$,

$$U_1^{(k)}(y) = -\xi^{(k)}(y_3)y_2 + \eta_1^{(k)}(y_3), \quad U_2^{(k)}(y) = \xi^{(k)}(y_3)y_1 + \eta_2^{(k)}(y_3) \quad (i=1,2).$$

From the boundary condition, we get

$$U_1^{(k)}(y) = \eta_1^{(k)}(y_3), \quad U_2^{(k)}(y) = \eta_2^{(k)}(y_3), \quad U_3^{(k)}(y) = \eta_3^{(k)}(y_3) - y_1 \frac{d\eta_1^{(k)}}{dy_3} - y_2 \frac{d\eta_2^{(k)}}{dy_3}.$$

$$\kappa_{33} = E_{33}(U^{(k)}) = \frac{d\eta_3^{(k)}}{dy_3} - y_1 \frac{d^2\eta_1^{(k)}}{dy_3^2} - y_2 \frac{d^2\eta_2^{(k)}}{dy_3^2}.$$

Eventually, it follows

$$\int_\Omega Y \left(\frac{d\eta_3^{(k)}}{dy_3} - y_1 \frac{d^2\eta_1^{(k)}}{dy_3^2} - y_2 \frac{d^2\eta_2^{(k)}}{dy_3^2} \right) \left(\frac{d\rho_3}{dy_3} - y_1 \frac{d^2\rho_1}{dy_3^2} - y_2 \frac{d^2\rho_2}{dy_3^2} \right) dy$$
$$= \widetilde{\Lambda}_k \int_\Omega \left(\eta_1^{(k)} \rho_1 + \eta_2^{(k)} \rho_2 \right) dy.$$

for any test functions ρ_1, ρ_2, ρ_3 (b.c. on $y_3 = 0, \ell$).

$\eta_3^{(k)} \equiv 0$ and for $i = 1, 2$, we have

$$\frac{Y}{4}\frac{d^4\eta_i^{(k)}}{dy_3^4} - \widetilde{\Lambda}\eta_i^{(k)} = 0 \quad (0 < y_3 < \ell)$$

and the boundary condition

$$\eta_i^{(k)}(0) = (d\eta_i^{(k)}/dy_3)(0) = 0, \quad \eta_i^{(k)}(\ell) = (d\eta_i^{(k)}/dy_3)(\ell) = 0.$$

Take the system of the eigenvalues $\{\Lambda_k\}_{k=1}^{\infty}$ of

$$(LEP) \qquad \begin{cases} \dfrac{Y}{4}\dfrac{d^4\eta_i^{(k)}}{d\tau^4} - \widetilde{\Lambda}\eta_i^(k) = 0 \quad (0 < \tau < \ell) \\ \eta_i^{(k)}(0) = (d\eta_i^{(k)}/dy_3)(0) = 0, \quad \eta_i^{(k)}(\ell) = (d\eta_i^{(k)}/dy_3)(\ell) = 0. \end{cases}$$

We have $\widetilde{\Lambda}_k \geqq \Lambda_k \quad (k \geqq 1)$.

(II) To prove the converse inequality $\widetilde{\Lambda}_k \leqq \Lambda_k$, we estimate $\mu_k(\epsilon)/\epsilon^2$ from above, with the aid of more precise test function. Take the corresponding eigenfunction

$$\Theta_k(y) = (\eta_1^{(k)}, \eta_2^{(k)}, -y_1(d\eta_1^{(k)}/d\tau)(y_3) - y_2(d\eta_1^{(k)}/d\tau)(y_3))$$

of (LEP).

By this function, we try the following test function in the functional $\widetilde{R}_\epsilon$.

$$\Theta_\epsilon(y) = (\eta_1^{(k)}, \eta_2^{(k)}, -y_1(d\eta_1^{(k)}/d\tau)(y_3) - y_2(d\eta_1^{(k)}/d\tau)(y_3)) + (\epsilon^2\phi_1(y), \epsilon^2\phi_2(y), \phi_3(y)).$$

Put $N = d\eta_3/dy_3 - y_1(d^2\eta_1^{(k)}/dy_3^2) - y_2(d^2\eta_2^{(k)}/dy_3^2)$ and calculate

$$\widetilde{R}_\epsilon(\Theta_\epsilon) = \frac{\int_\Omega \left(\lambda_1 \left(\epsilon^2 \frac{\partial \phi_1}{\partial y_1} + \epsilon^2 \frac{\partial \phi_2}{\partial y_2} + \epsilon^2 N + \epsilon^3 \frac{\partial \phi_3}{\partial y_3} \right)^2 + 2\lambda_2 \left(\sum_{i,j=1}^2 \epsilon^4 E_{ij}(\phi)^2 \right) \right) dy}{\int_\Omega \left(\epsilon^2(\eta_1 + \epsilon^2\phi_1)^2 + \epsilon^2(\eta_2 + \epsilon^2\phi_2)^2 + \epsilon^4(\eta_3 - y_1 \frac{d\eta_1^{(k)}}{dy_3} - y_2 \frac{d\eta_2^{(k)}}{dy_3} + \epsilon\phi_3)^2 \right) dy}$$

$$+ \frac{\int_{\Omega)} 2\lambda_2 \left(2\epsilon^2 \sum_{i=1}^2 \frac{1}{4} \left(\epsilon^2 \frac{\partial \phi_i}{\partial y_3} + \epsilon \frac{\partial \phi_3}{\partial y_i} \right)^2 + \epsilon^4 \left(N + \epsilon \frac{\partial \phi_3}{\partial y_3} \right)^2 \right) dy}{\int_\Omega \left(\epsilon^2(\eta_1 + \epsilon^2\phi_1)^2 + \epsilon^2(\eta_2 + \epsilon^2\phi_2)^2 + \epsilon^4(\eta_3 - y_1 \frac{d\eta_1^{(k)}}{dy_3} - y_2 \frac{d\eta_2^{(k)}}{dy_3} + \epsilon\phi_3)^2 \right) dy}.$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\widetilde{R}_\epsilon(\Theta_\epsilon)=\frac{\int_\Omega \lambda_1\left(\frac{\partial\phi_1^{(k)}}{\partial y_1}+\frac{\partial\phi_2^{(k)}}{\partial y_2}+N\right)^2 dy}{\int_\Omega((\eta_1^{(k)})^2+(\eta_2^{(k)})^2)dy}$$
$$+\frac{\int_\Omega 2\lambda_2\left(\sum_{i,j=1}^2 E_{ij}(\phi)^2+\frac{1}{2}\sum_{i=1}^2\left(\frac{\partial\phi_3}{\partial y_i}\right)^2+N^2\right)dy}{\int_\Omega((\eta_1^{(k)})^2+(\eta_2^{(k)})^2)dy}.$$

We want to make the right hand side value as lower as possible. Seek for (ϕ_1,ϕ_2,ϕ_3) in the following form

$$\phi_i(y)=\sum_{p,q=1}^2\alpha_{pq}^{(i)}y_py_q+\sum_{p=1}^2\beta_p^{(i)}y_p \qquad (i=1,2),$$
$$\phi_3(y)=0$$

where $\alpha_{pq}^{(i)}$ and $\beta_p^{(i)}$ depend only on y_3 determined in next page.

$$\alpha_{11}^{(1)} = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{d^2\eta_1^{(k)}}{dy_3^2},\ \alpha_{12}^{(1)} = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{d^2\eta_2^{(k)}}{dy_3^2},\ \alpha_{22}^{(1)} = -\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{d^2\eta_1^{(k)}}{dy_3^2}$$
$$\alpha_{11}^{(2)} = -\frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{d^2\eta_2^{(k)}}{dy_3^2},\ \alpha_{12}^{(2)} = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{d^2\eta_1^{(k)}}{dy_3^2},\quad \alpha_{22}^{(2)} = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{d^2\eta_2^{(k)}}{dy_3^2}$$
$$\beta_1^{(1)} = 0,\ \beta_2^{(2)} = 0$$

and accordingly

$$\phi_1(y) = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(\frac{d^2\eta_1^{(k)}}{dy_3^2}y_1^2 + 2\frac{d^2\eta_2^{(k)}}{dy_3^2}y_1y_2 - \frac{d^2\eta_1^{(k)}}{dy_3^2}y_2^2\right),$$
$$\phi_2(y) = \frac{1}{4}\frac{\lambda_1}{\lambda_1+\lambda_2}\left(-\frac{d^2\eta_2^{(k)}}{dy_3^2}y_1^2 + 2\frac{d^2\eta_1^{(k)}}{dy_3^2}y_1y_2 + \frac{d^2\eta_2^{(k)}}{dy_3^2}y_2^2\right),$$
$$\phi_3(y) = 0.$$

We calculate and estimate $\limsup_{\epsilon\to 0} \widetilde{R}_\epsilon(\Theta_\epsilon)/\epsilon^2$ with Max-Min principle argument and get eventually

$$\widetilde{\Lambda}_k \leqq \Lambda_k.$$

This result concludes the proof of

$$\lim_{\epsilon\to 0} \frac{\mu_k(\epsilon)}{\epsilon^2} = \Lambda_k.$$

See S.J., A. Rodrïguez Mulet [14] for the details of proof for general cases.

§3. Middle eigenvalues - Stretching, Torsion mode

We study the eigenvalues which are away from zero.

$a = a(x_3)$: C^3- positive function in $[0, \ell]$

$$\Omega_a(\epsilon) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 < x_3 < \ell, x_1^2 + x_2^2 < \epsilon^2 a(x_3)^2$$
$$\Gamma_{a,1}(\epsilon) = \{x \in \partial\Omega_a(\epsilon) \mid x_3 = 0 \,\text{or}\, \ell\}$$
$$\Gamma_{a,2}(\epsilon) = \{x \in \partial\Omega_a(\epsilon) \mid 0 < x_3 < \ell\}$$

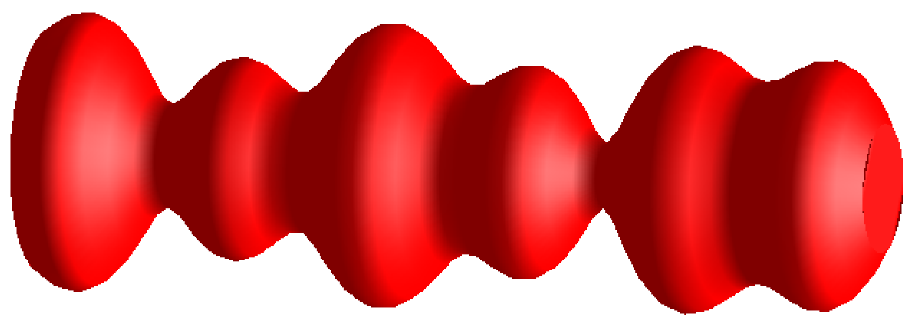


FIGURE 4. Example of $\Omega_a(\varepsilon)$

Eigenvalue problem

$$(EP) \qquad \begin{cases} L[\Phi] + \mu\,\Phi = \mathbf{0} \quad \text{in} \quad \Omega_a(\epsilon) \\ \Phi = \mathbf{0} \quad \text{on} \quad \Gamma_{1,a}(\epsilon) \\ \sigma(\Phi)\nu = \mathbf{0} \quad \text{on} \quad \Gamma_{2,a}(\epsilon) \end{cases}$$

We can deal with solutions of special type due to the axissymmetry of the domain.

$$\Phi(x) = \begin{pmatrix} -x_2\rho(s,x_3) \\ x_1\rho(s,x_3) \\ 0 \end{pmatrix} + \begin{pmatrix} x_1\chi(s,x_3) \\ x_2\chi(s,x_3) \\ \tau(s,x_3) \end{pmatrix}$$

where $s = (x_1^2 + x_2^2)^{1/2}$.

Idea : Let $\mathcal{X}(\epsilon) = L^2_*(\Omega_a(\epsilon);\mathbb{R}^3)$ be the subspace generated by the above functions. Due to the symmetry of the domain, this subspace is invariant w.r.t. the operator L.

$\{\mu_{*,k,a}(\epsilon)\}_{k=1}^{\infty}$: eigenvalues of (EP) in $\mathcal{X}(\epsilon) \Longleftarrow$ how to detect !

Theorem (S.J., A. Rodríguez Mulet). There exist subclasses of eigenvalues $\{\mu^S_{k,a}(\epsilon)\}_{k=1}^{\infty}$ and $\{\mu^T_{k,a}(\epsilon)\}_{k=1}^{\infty}$ such that

(i)

$$\mu^S_{k,a} := \lim_{\epsilon\to 0} \mu^S_{k,a}(\epsilon) \quad \text{(exists)}$$

$$\mu^T_{k,a} := \lim_{\epsilon\to 0} \mu^T_{k,a}(\epsilon) \quad \text{(exists)}$$

(ii) $\mu^T_{k,a}$ agrees to the $k-$the eigenvalue of the ODE eigenvalue problem

$$\lambda_2 \frac{d}{dy_3}\left(a(y_3)^4 \frac{d\rho}{dy_3}\right) + \mu\, a(y_3)^4 \rho = 0 \quad (0 < y_3 < \ell), \qquad \rho(0) = 0, \quad \rho(\ell) = 0$$

(iii) $\mu^S_{k,a}$ agrees to the $k-$the eigenvalue of the ODE eigenvalue problem

$$Y \frac{d}{dy_3}\left(a(y_3)^2 \frac{d\tau}{dy_3}\right) + \mu\, a(y_3)^2 \tau = 0 \quad (0 < y_3 < \ell), \qquad \tau(0) = 0, \quad \tau(\ell) = 0$$

その他のトピックス

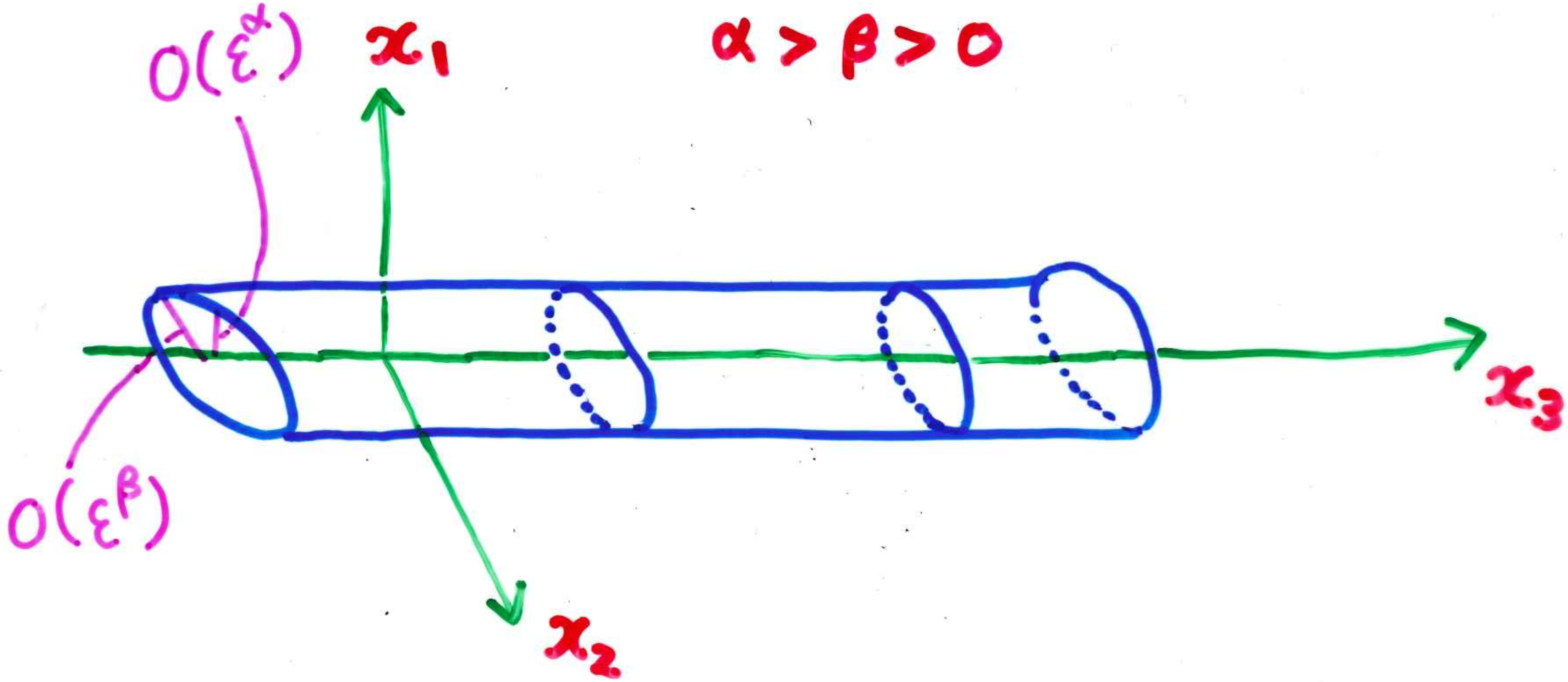
S.J, E. Ushikoshi, H. Yoshihara, Asymptotic behavior of the eigenfrequencies of a thin elastic rod with non-uniform cross-section for non-isotropic shrinking, preprint,　がある.

弾性体が一方方向につぶれている場合にどうなるか？ 断面(小さい 2 次元集合)が x_1 方向と, x_2 方向の微小のオーダーが異なる(アスペクト比が極端). 次の領域が典型例.

$\alpha > \beta > 0$: パラメータ

$$\Omega(\epsilon) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \frac{x_1^2}{\epsilon^{2\alpha}} + \frac{x_2^2}{\epsilon^{2\beta}} \leqq 1,\ 0 < x_3 < \ell\right\}$$

x_1 方向に曲げがあるモードが一番小さいクラスの固有値に相当する.



Thin Rod

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