走化性方程式の爆発解の構成 I

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Based on the paper by Mizoguchi-S. (2007)

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Our system and situation

$$(PE)\begin{cases} u_t = \nabla\cdot(\nabla u - u\nabla v) \text{ in } \mathbf{R}^N \times [0,T), \\ 0 = \Delta v + u \text{ in } \mathbf{R}^N \times [0,T), \\ u(\cdot,0) = u_0 \text{ in } \Omega. \end{cases}$$

- $N \geq 3$. $u_0 \geq 0$ is radial. $\|u_0\|_{\infty,1} < \infty$, where $\|f\|_{\infty,1} = \sup_{x\in\mathbf{R}^N} (1+|x|^2)|f(x)|$.
- $\exists T > 0$ s.t.

$$\left(\begin{array}{l} \exists_1 u \in C([0,T) : L_1^\infty), u \text{ is positive and radial,} \\ \nabla v(x,t) = \dfrac{x}{\omega_N |x|^N} \displaystyle\int_{|\tilde{x}|<|x|} u(\tilde{x},t)d\tilde{x}. \end{array} \right.$$

Our system and situation

• We say that a solution to (PE) blows up at $t = T$, if u is bounded in $\mathbf{R}^N$ for $t \in (0, T)$, and satisfies

$$\boxed{\lim_{t \to T} \sup_{x \in \mathbf{R}^N} u(x,t) = \infty.}$$

• We say that a solution u to (PE) is a (backward) self-similar solution, if there exist a function $\overline{u}$ in $\mathbf{R}^N$ and a positive constant T satisfying

$$\boxed{u(x,t) = \frac{1}{T-t}\overline{u}\left(\frac{x}{\sqrt{T-t}}\right).}$$

Then, we say that $\overline{u}$ is a profile function.

Our system and situation

We say that the blowup is of **Type I and Type II**, if a solution u to (PE) blowing up at $T \in (0, \infty)$ satisfies

$$\limsup_{t \nearrow T} (T-t) \sup_{x \in \mathbf{R}^N} u(x,t) < \infty \text{ and } = \infty, \text{ respectively.}$$

In the case where $N \geq 3$,

$$\overline{u}(x) = \frac{8N(N-2) + 4(N-1)|x|^2}{[2(N-2) + |x|^2]^2}$$

is a profile function. Then, for $T > 0$
$u(x,t) = \overline{u}(x/\sqrt{T-t})/(T-t)$ is a self-similar solution satisfying the following.

Our system and situation

- $\lim_{t\to\infty}(T-t)\sup_{x\in\mathbf{R}^N} u(x,t) = \sup_{x\in\mathbf{R}^N} \overline{u}(x)$.
- $$\lim_{t\to T} u(x,t) = \lim_{t\to T} \frac{1}{|x|^2}\cdot\left(\frac{|x|^2}{T-t}\overline{u}\left(\frac{|x|}{\sqrt{T-t}}\right)\right)$$
  $$= \frac{4(N-1)}{|x|^2}$$ loc. unif. in $\mathbf{R}^N \setminus \{0\}$.

Our system and situation

Theorem 1 (Mizoguchi-S.,'07)
Let $N \geq 11$ and $\nu = \{-(N+2) + \sqrt{(N-10)(N-2)}\}/4$.
For $0 < T \ll 1$ and $\forall J \geq 2$, there exists a positive and radial solution u to (PE) exhibiting Type II blowup at T.
Moreover, the solution u satisfies

$$\lim_{t \to T} (T-t)^{\boxed{-J/(\nu+1)}} \sup_{x \in \mathbf{R}^N} u(x,t) \in (0,\infty),$$

$u(x,T) = O(1/|x|^2)$ as $|x| \to 0$.

- $J \geq 2$, $\nu \in [-5/2, -2)$,

$\boxed{J/(\nu+1) \leq -4/3}$. **(<u>Type II blowup</u>)**

Proof of Theorem 1 ∼ Rescaling ∼

For a radial solution u, putting

$$M(r,t) = \frac{1}{\omega_N r^N} \int_{|x|<r} u(x,t)\,dx,$$

$$\Psi(y,\tau) = (T-t)M(r,t), \quad y = r/\sqrt{T-t},$$

$$\tau = -\log(T-t), \quad \tau_0 = -\log T,$$

it holds that

$$\mathcal{M}(\Psi) = \Psi_\tau - \Psi_{yy} - \left(\frac{N+1}{y} - \frac{y}{2}\right)\Psi_y + \Psi - \Psi\left(y\Psi_y + N\Psi\right).$$

$\#$ We can use the comparison theorem.

Proof of Theorem 1 $\sim$ Rescaling $\sim$

Putting $\Psi_\infty(y) = 2/y^2$, it holds $\mathcal{M}(2/y^2) = 0$.
Then, Ψ_∞ is a singular stationary solution.

$$
\begin{aligned}
\mathcal{M}(2/y^2) &= (2/y^2)_\tau - (2/y^2)_{yy} - \left(\frac{N+1}{y} - \frac{y}{2}\right)(2/y^2)_y \\
&\quad + (2/y^2) - (2/y^2)\left\{y(2/y^2)_y + N(2/y^2)\right\} \\
&= -(12/y^4) - \left(\frac{N+1}{y} - \frac{y}{2}\right)(-4/y^3) \\
&\quad + (2/y^2) - \{N-2\}(4/y^4) \\
&= 0.
\end{aligned}
$$

Proof of Theorem 1 $\sim$ Rescaling $\sim$

Putting $\phi = \Psi - \Psi_\infty$, ϕ satisfies

$\mathcal{N}(\phi) = \phi_\tau + A\phi - F(\cdot, \phi)$,

$A\phi = -\phi_{yy} - \left(\frac{N+3}{y} - \frac{y}{2}\right)\phi_y + \phi - \frac{4(N-1)}{y^2}\phi$,

$F(y, \phi) = N\phi^2 + y\phi\phi_y$.

In fact,

$$\begin{aligned} \mathcal{M}(\phi+\Psi_\infty) &= (\phi+\Psi_\infty)_\tau - (\phi+\Psi_\infty)_{yy} - \left(\frac{N+1}{y} - \frac{y}{2}\right)(\phi+\Psi_\infty)_y \\ &\quad +(\phi+\Psi_\infty) - (\phi+\Psi_\infty)\{y(\phi+\Psi_\infty)_y + N(\phi+\Psi_\infty)\} \\ &= \mathcal{M}(\phi) + \mathcal{M}(\Psi_\infty) \\ &\quad -\Psi_\infty\{y\phi_y + N\phi\} - \phi\{y(\Psi_\infty)_y + N\Psi_\infty)\} \\ &= \mathcal{M}(\phi) - \frac{2}{y}\phi_y - \frac{4(N-1)}{y^2}\phi = A\phi + F(y, \phi). \end{aligned}$$

Proof of Theorem 1 ∼ Rescaling ∼

A is self-adjoint in

$L^2_w = \left\{ f \in L^2_{loc}(\mathbf{R}_+) : \|f\|^2 = \int_0^\infty |f(y)|^2 y^{N+3} e^{-y^2/4} dx < \infty \right\}$.

In fact, $A\phi = -\phi_{yy} - \left(\frac{N+3}{y} - \frac{y}{2}\right)\phi_y + \phi - \frac{4(N-1)}{y^2}\phi$,

$$\langle A\phi, \psi \rangle = \int_0^\infty \left(\phi_y \psi_y + \phi\psi - \frac{4(N-1)}{y^2}\phi\psi \right) y^{N+3} e^{-y^2/4} dy.$$

The eigen values and eigen functions are

$\{\lambda_j\}_{j=0}^\infty = \{j+1+\nu\}_{j=0}^\infty$ and

$\{\varphi_j(y)\} = \{c_j y^{2\nu} L_j^\alpha(y^2/4)\}$, respectively.

Here L_j^α is an associated Laguerre polynomial with

$L_j^\alpha(0) > 0$, and c_j is determined so that $\|\varphi_j\| = 1$.

Strategy of proof

$\#\ \lambda_j = j + 1 + \nu > 0$ ($j \geq 2$).
In fact, $N \geq 11$ and $\nu = \{-(N+2) + \sqrt{(N-10)(N-2)}\}/4$.

$$\begin{aligned} \nu &= \frac{-(N+2)^2 + (N-10)(N-2)}{4\{(N+2) + \sqrt{(N-10)(N-2)}\}} \\ &= \frac{-16N + 16}{4\{(N+2) + \sqrt{(N-10)(N-2)}\}} \to -2 \text{ as } N \to \infty. \end{aligned}$$

$\nu_{|N=11} = -5/2 \leq \nu < -2 = \nu_{|N=\infty}$.

- Let $J \geq 2$, $\eta = \lambda_J/(-2\nu - 1)$, $\tau_0 \gg 1$,
  $0 < (1/2) - \sigma \ll 1$, $K \gg 1$.
  $\gamma_J = \lim_{y\to 0} y^{-2\nu}\varphi_J(y) = c_J L_J^{\alpha}(0) > 0$.

Strategy of proof

• We divide $[0, \infty)$ into three regions;
Inner region $[0, Ke^{-\eta\tau}]$.
Middle region $[Ke^{-\eta\tau}, e^{\sigma\tau}]$.
Outer region $[e^{\sigma\tau}, \infty)$.

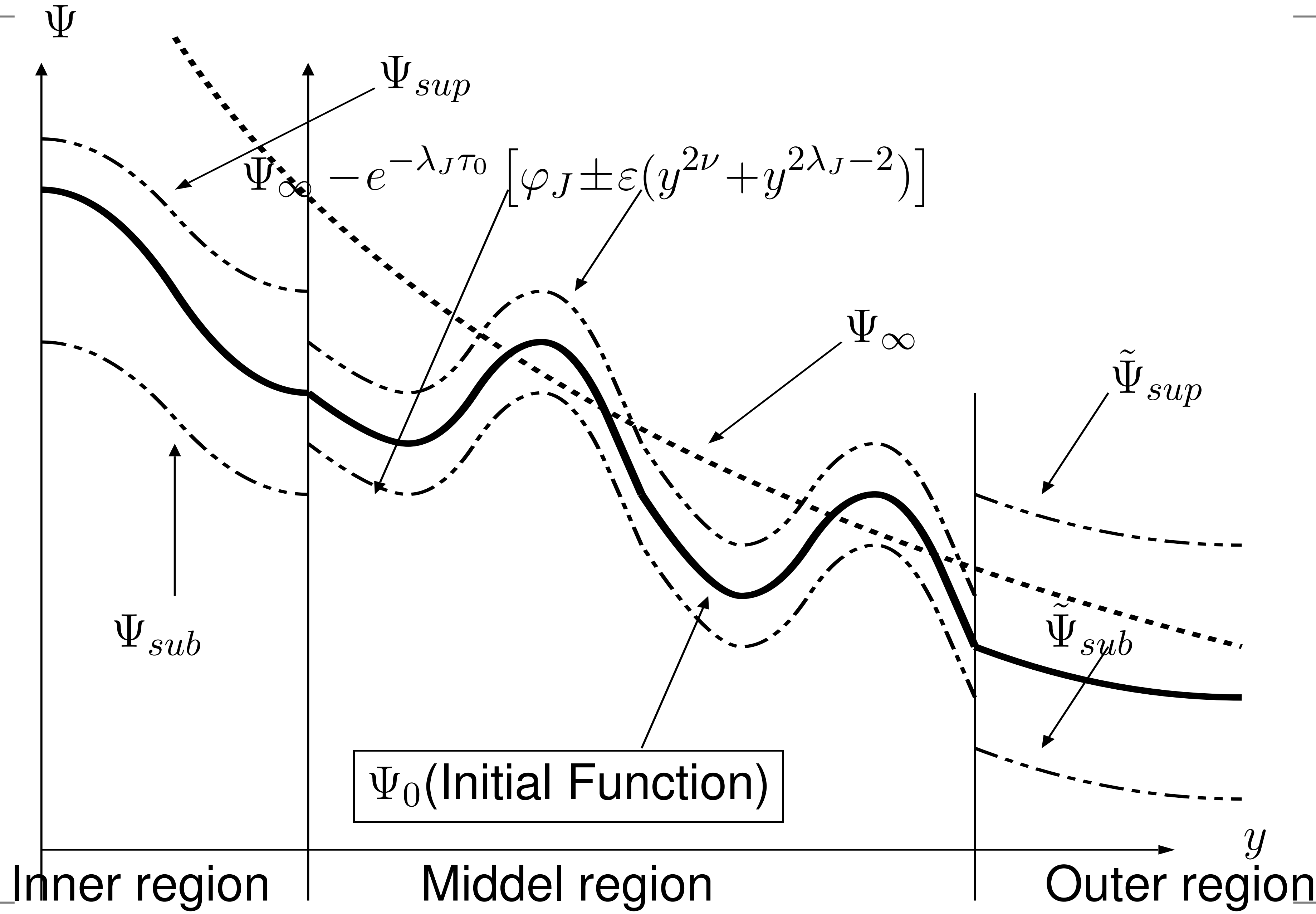
• In the middle region, by choosing suitable Fourier coefficients d_0, d_1, $\cdots$, d_{J-1}, we construct suitable initial function $\Psi(y, \tau_0)$ such that

$$\Psi_0(y) = \Psi(y, \tau_0) = \Psi_\infty(y) + \sum_{j=0}^{J-1} d_j \varphi_j - e^{-\lambda_J \tau_0} \varphi_J,$$

$$\Psi(y, \tau) - \Psi_\infty(y) = O(1) e^{-\lambda_J \tau_0} \varphi_J \quad \text{as } \tau \to \infty.$$

• In the inner region and the outer region, by using the subsolution and the supersolution, we control the solution.

Initial function Ψ_0



Inner region

- Stationary solution.

Let $v = v(r, \ell)$ be the solution to the following problem;

$$-v_{rr} - \frac{N-1}{r} v_r = e^v, \;\; u = e^v \;\; \text{in } (0, \infty)$$

with $v(0, \ell) = \ell > 0$ and $v_r(0, \ell) = 0$.

Then, $(u, v) = (e^{v(\cdot,\ell)}, v(\cdot, \ell))$ be a stationary solution, since $0 = \Delta u - \nabla u \cdot \nabla v = \nabla \cdot u \nabla(\log u - v)$ and $0 = \Delta v + u$,

- Properties in the case where $N \geq 11$.

If $0 < \ell_1 < \ell_2$, then the solutions v_i whose initial data $v_i(0) = \ell_i$ satisfy $v_1 < v_2$ in $[0, \infty)$.

$$u(r, \ell) = \frac{2(N-2)}{r^2} - \gamma \ell^{-2(1+\nu)}(1 + o(1)) r^{2\nu} \;\; \text{as} \;\; r \to \infty$$

with some $\gamma > 0$.

Inner region

$\#$ Singular stationary solution.

$(u_\infty, v_\infty) = (2(N-2)/r^2, -2\log r)$ is a singular stationary solution to the original system.
$\Psi_\infty(r) = 2/r^2 = r^{-N}\int_0^r u_\infty(\xi)\xi^{N-1}d\xi$ is a singular stationary solution to $\mathcal{M}(\Psi) = 0$.

• Let $0 < \varepsilon \ll 1$. We chose ℓ_1 and ℓ_2 with $0 < \ell_1 < \ell_2$ such that

$$u_1(y) = u(y, \ell_1) = \frac{2(N-2)}{r^2} - (\gamma_J + 3\varepsilon)(1 + o(1))y^{2\nu}$$
$$u_2(y) = u(y, \ell_2) = \frac{2(N-2)}{r^2} - (\gamma_J - 3\varepsilon)(1 + o(1))y^{2\nu}$$

as $y \to \infty$.
Here, $\varphi_J(y) \sim \gamma_J y^{2\nu}$ as $y \sim Ke^{-\eta\tau} \sim 0$ and $\tau \gg 1$.

Inner region

- Let $\chi = (2\nu+1)/(2\nu+2) > 1$. ($0 < \varepsilon \ll 1$, $K \gg 1$, $\eta = \lambda_J/(-2\nu-1) > 0$.) We define Ψ_{sub} and Ψ_{sup} as the following;

$$\Psi_{sub}(y,\tau) = (1+\varepsilon K^{2\nu+2}e^{-\eta\tau})\frac{e^{2\chi\eta\tau}}{(e^{\chi\eta\tau})^N}\int_0^{e^{\chi\eta\tau}y} u_1(\xi)\xi^{N-1}d\xi,$$

$$\Psi_{sup}(y,\tau) = (1-\varepsilon K^{2\nu+2}e^{-\eta\tau})\frac{e^{2\chi\eta\tau}}{(e^{\chi\eta\tau})^N}\int_0^{e^{\chi\eta\tau}y} u_2(\xi)\xi^{N-1}d\xi.$$

Then, the following hold in the inner region;

$0 < \Psi_{sub} < \Psi_{sup}$;
$\mathcal{M}(\Psi_{sub}) < 0$ and $\mathcal{M}(\Psi_{sup}) > 0$.

Inner region

- Inner region: $[0, Ke^{-\eta\tau}]$, $\tau \geq \tau_0 \gg 1$

$\Psi_{sub}(y,\tau) < \Psi_{sup}(y,\tau)$.

Ψ_{sub} is a subsolution to $\mathcal{M}(\Psi) = 0$.

Ψ_{sup} is a supersolution to $\mathcal{M}(\Psi) = 0$.

$\Psi_{sup}(y,\tau) \geq \Psi_\infty(y) - (\gamma_J - 2\varepsilon)e^{-\lambda_J\tau}y^{2\nu}$ for $y = O(e^{-\eta\tau})$,

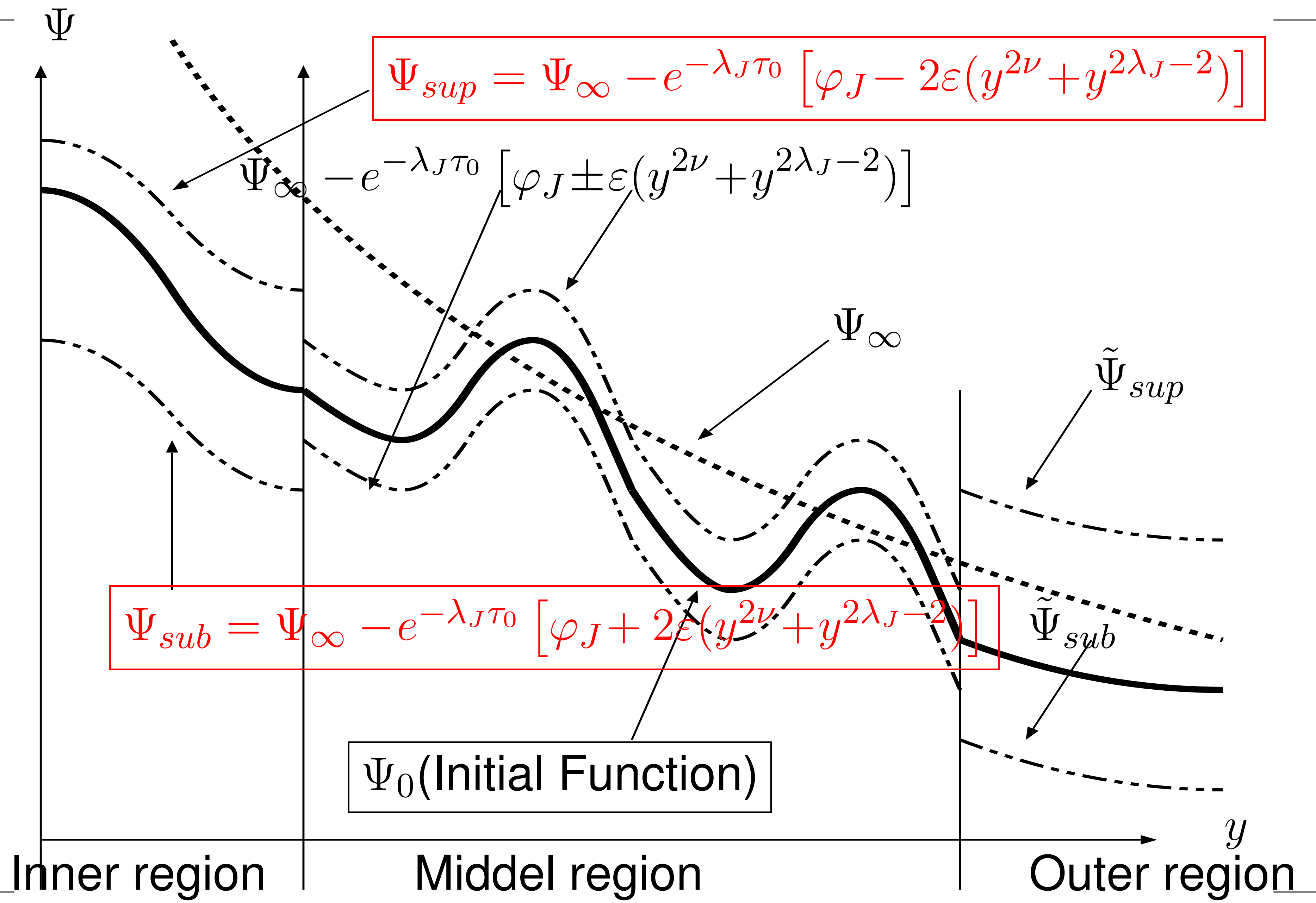
$\Psi_{sub}(y,\tau) \leq \Psi_\infty(y) - (\gamma_J + 2\varepsilon)e^{-\lambda_J\tau}y^{2\nu}$ for $y = O(e^{-\eta\tau})$.

Then, we define the initial function

$\Psi_0(y) = \theta_{\mathcal{I}}\Psi_{sub}(y,\tau_0) + (1-\theta_{\mathcal{I}})\Psi_{sup}(y,\tau_0)$ with $\theta_{\mathcal{I}} \in (0,1)$

such that Ψ_0 is continuous at the boundary between the inner region and the middle region.

Initial function Ψ_0



Outer region

- $\Psi_{SE}(y) = 4/[2(N-2)+y^2]$ satisfies $\mathcal{M}(\Psi_{SE}) = 0$.

Ψ_{SE} corresponds to the backward self-similar solution.

$$u(x,t) = \frac{1}{T-t}\overline{u}\left(\frac{x}{\sqrt{T-t}}\right),$$

$$\overline{u}(x) = \frac{8N(N-2)+4(N-1)|x|^2}{[2(N-2)+|x|^2]^2}$$

- Outer region: $[e^{\sigma\tau}, \infty)$, $\tau \geq \tau_0$.

$\tilde{\Psi}_{sup} = \frac{2+\varepsilon}{4}\Psi_{SE}$ is a supersolution.

$\tilde{\Psi}_{sub} = \Psi_{\infty} - \frac{\varepsilon}{2y^2}\exp\left(\frac{N-2}{\sigma}[e^{-2\sigma\tau_0} - e^{-2\sigma\tau}]\right)$ is a subsolution.

Outer region

- Outer region: $[e^{\sigma\tau}, \infty)$, $\tau \geq \tau_0$.

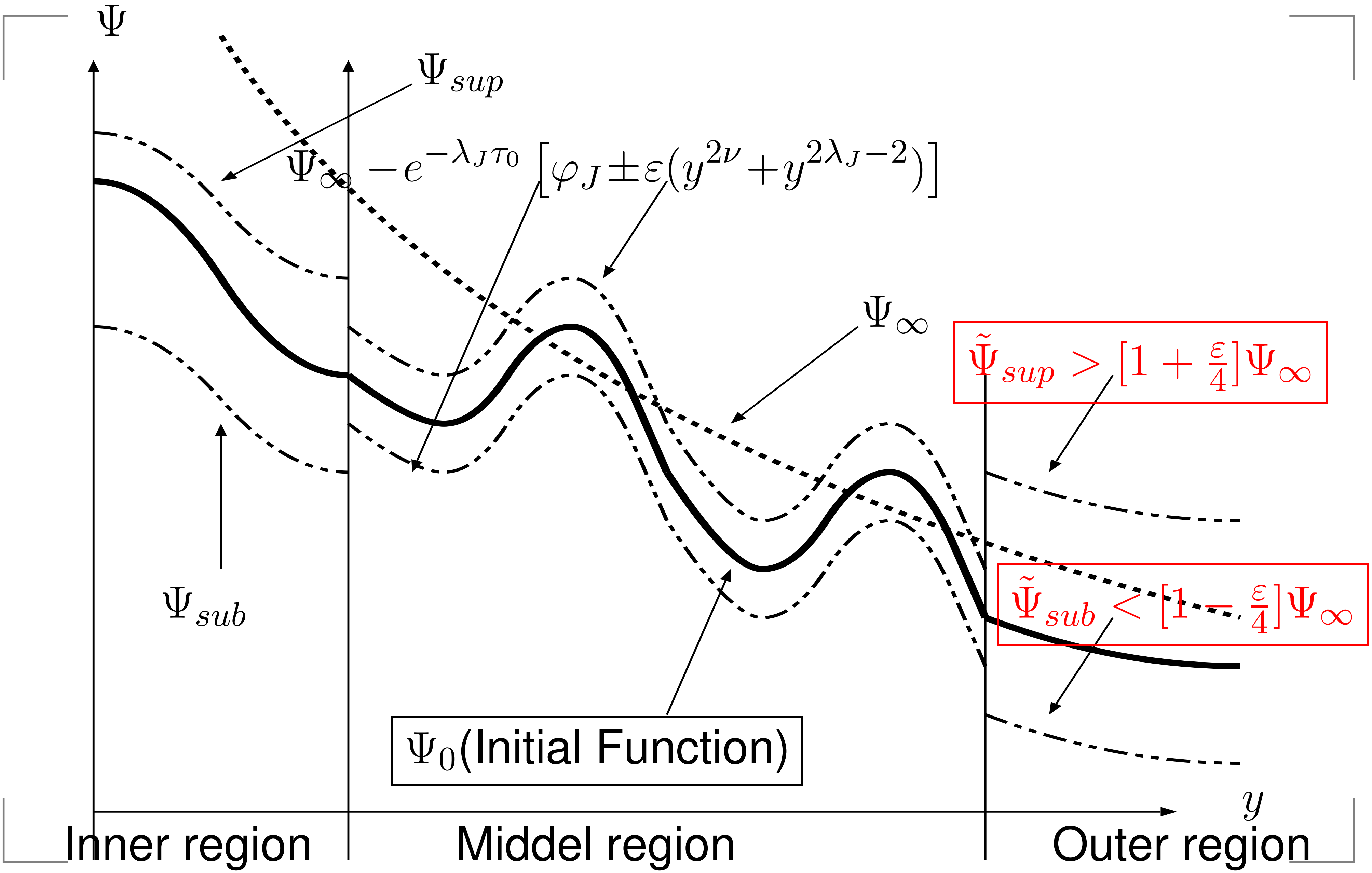
$[1-\varepsilon]\Psi_\infty(y) < \tilde{\Psi}_{sub}(y,\tau) < [1-(\varepsilon/4)]\Psi_\infty(y)$,

$[1+(\varepsilon/4)]\Psi_\infty(y) < \tilde{\Psi}_{sup}(y,\tau) < [1+\varepsilon]\Psi_\infty(y)$.

Then, we define the initial function

$\Psi_0(y) = \theta_{\mathcal{O}}\Psi_{sub}(y,\tau_0) + (1-\theta_{\mathcal{O}})\Psi_{sup}(y,\tau_0)$, where $\theta_{\mathcal{O}} \in (0,1)$ such that Ψ_0 is continuous at the boundary between the middle region and the outer region.

Initial function Ψ_0



Middel region

- Middel region: $[Ke^{-\eta\tau}, e^{\sigma\tau}]$, $\tau \geq \tau_0$

  $\Psi_0(y) = \Psi_\infty + \sum_{j=0}^{J-1} d_j \varphi_j - e^{-\lambda_J \tau_0} \varphi_J$.

  $\sum_{j=0}^{J-1} |d_j| < \varepsilon \theta_1 e^{-\lambda_J \tau_0}$.

$$\theta_1 = \left\{ \sum_{j=0}^{J-1} \left(\sup_{0<y<\infty} \frac{|\varphi_j(y)|}{y^{2\nu} + y^{2\lambda_J - 2}} \right) + 1 \right\}^{-1}.$$

$$|\Psi_0(y) - \Psi_\infty(y) + \gamma_J e^{-\lambda_J \tau_0} y^{2\nu}|$$
$$< \varepsilon e^{-\lambda_J \tau_0} y^{2\nu} \quad \text{for } y = O(e^{-\eta\tau_0}).$$

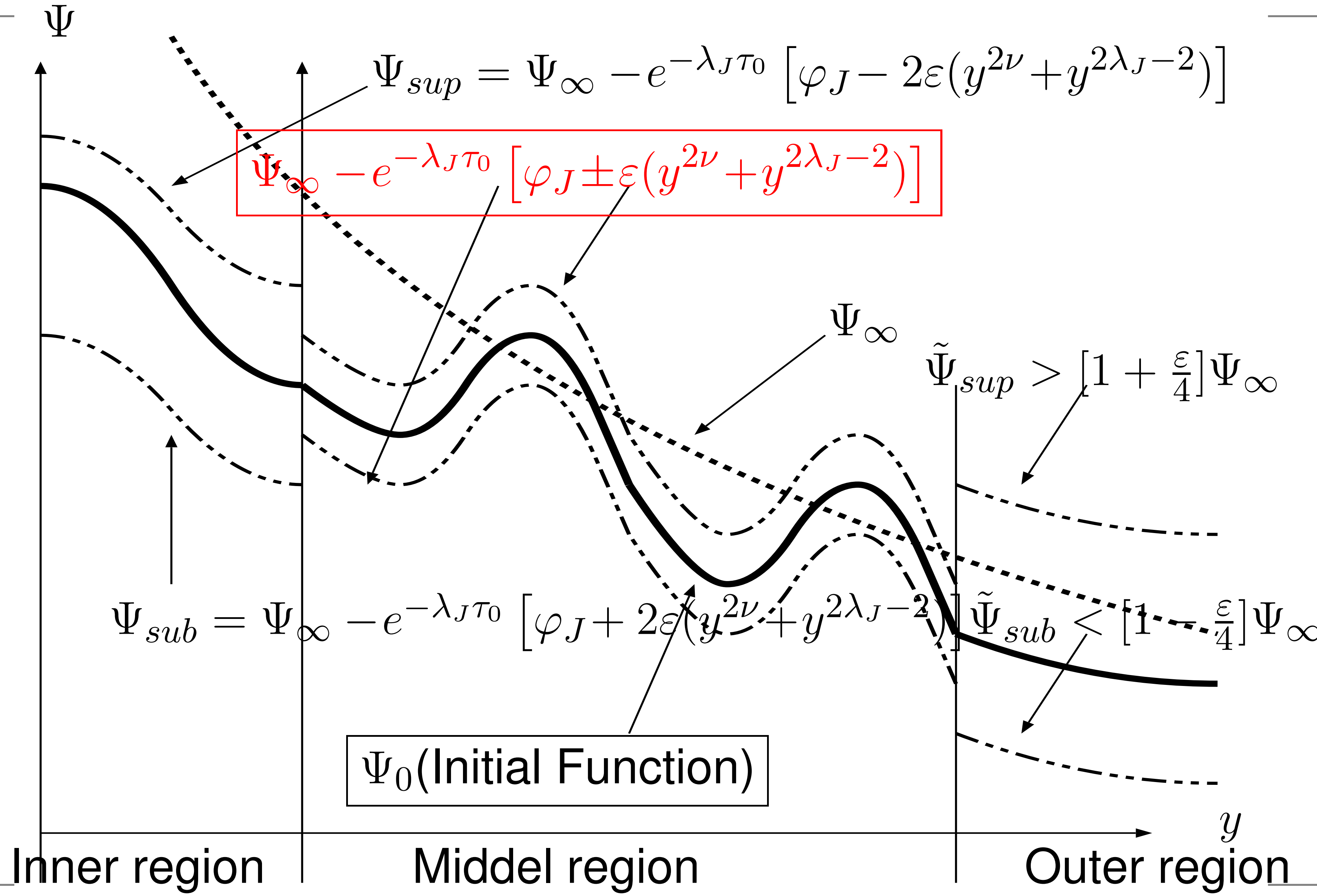
$$|\Psi_0(y) - \Psi_\infty(y)| < O(1) e^{-\lambda_J \tau_0} y^{2\lambda_J} y^{-2}$$
$$< Ce^{-(1-2\sigma)\lambda_J \tau_0} \Psi_\infty(y) < (\varepsilon/4) \Psi_\infty(y) \quad \text{for } y = O(e^{-\sigma\tau_0}).$$

※ $\varphi_J(y) = O(y^{2\lambda_J - 2})$ for $y \gg 1$.

※ $\varphi_J(y) \sim \gamma_J y^{2\nu}$ for $0 < y \ll 1$.

○ $\{d_j\}_{j=0}^{J-1} \underset{\text{unique}}{\Rightarrow} \{\theta_{\mathcal{I}}, \theta_{\mathcal{O}}\} \underset{\text{unique}}{\Rightarrow} \Psi_0(\cdot; d)$.

Initial function Ψ_0



Mapping P on Fourier coefficients

For $\tau_1 \geq \tau_0$ and $\theta \in (0,1]$, we define

$$\mathcal{A}(\tau_0,\tau_1;\theta) = \Big\{ h \in C([\tau_0,\tau_1]; L^\infty(\mathbf{R}_+)) :$$

$$\left| h(y,\tau) - \Psi_\infty(y) + e^{-\lambda_J \tau}\varphi_J(y) \right| < \theta\varepsilon e^{-\lambda_J\tau}\left(y^{2\nu} + y^{2\lambda_J - 2}\right)$$

$$\text{for } y \in [Ke^{-\eta\tau}, e^{\sigma\tau}] \text{ and } \tau \in [\tau_0,\tau_1] \Big\}.$$

If a solution $\Psi \in \mathcal{A}(\tau_0,\tau_1;\theta)$, for $\tau \in [\tau_0,\tau_1]$ Ψ satisfies

$\Psi_{sub} < \Psi < \Psi_{sup}$ in the inner region,
$\tilde{\Psi}_{sub} < \Psi < \tilde{\Psi}_{sup}$ in the outer region.

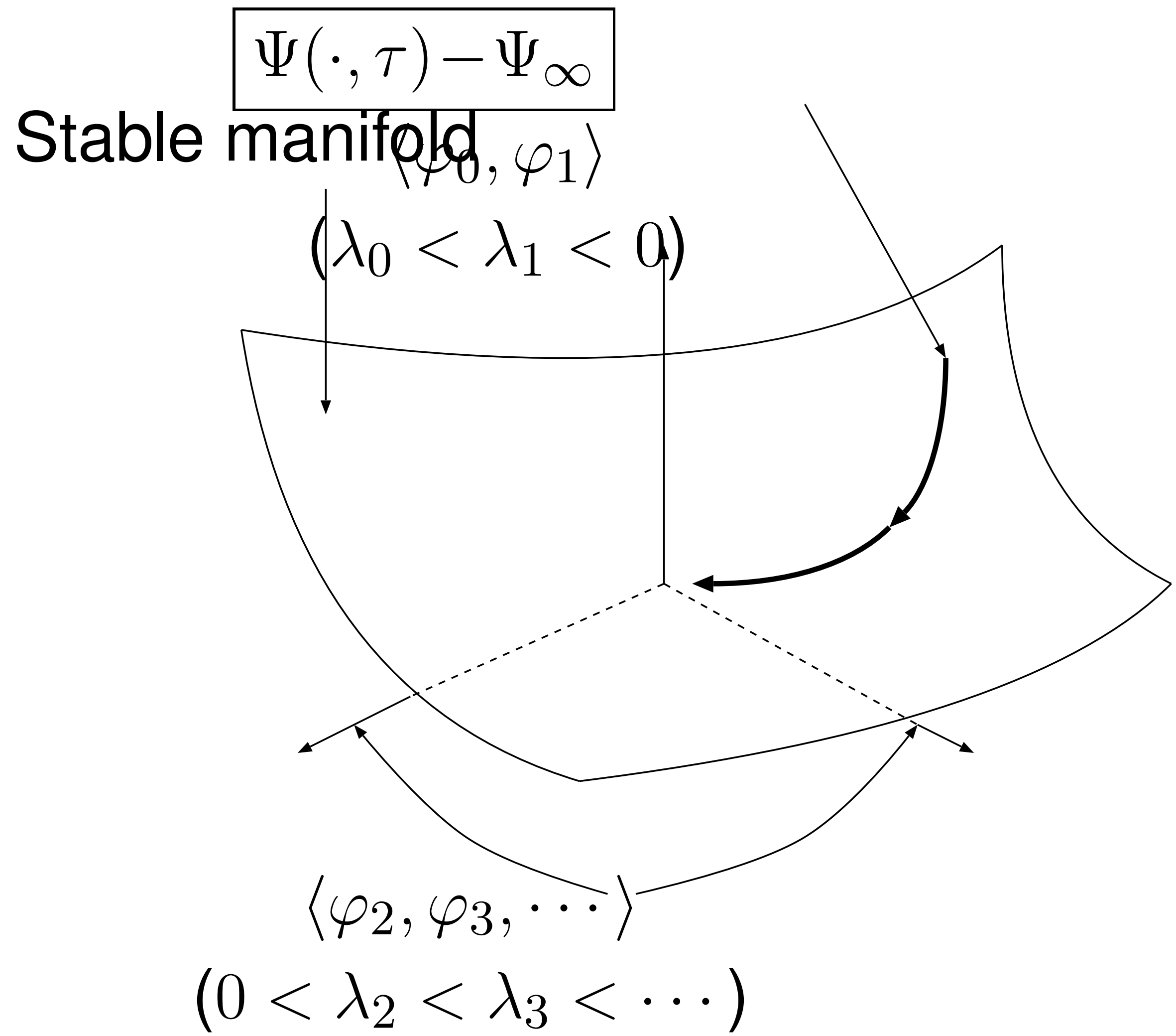
Mapping P on Fourier coefficients

We define
$\mathcal{U}(\tau_0,\tau_1) = \Big\{ d = (d_0, d_1, \cdots, d_{J-1}) \in \mathbf{R}^J :$
$\Psi \in \mathcal{A}(\tau_0,\tau_1,1)$ with $\Psi(\cdot,\tau_0) = \Psi_0(\cdot; d) \Big\}$.

We define
$P(d;\tau_0,\tau_1) = (p_0, p_1, \cdots, p_{J-1})$ as $p_j = \langle \Psi(\cdot,\tau_1) - \Psi_\infty, \varphi_j \rangle$
($j=0,1,\cdots,J-1$),
where $\Psi(\cdot,\tau_0) = \Psi_0(\cdot; d)$,

$$\langle f, g \rangle = \int_0^\infty f(y) g(y) y^{N+3} e^{-y^2/4} dx.$$

The way to decide $d \sim J = 2 \sim$



The way to decide $d \sim J = 2 \sim$

- d is determined so that $\Psi \in \overline{\mathcal{A}(\tau_0, \infty; 1)}$.

Then,

· the orbit $\{\Psi - \Psi_\infty\}$ is on the stable manifold.

· the main term of $\Psi - \Psi_\infty$ is $e^{-\lambda_2 \tau} \varphi_2$.

Properties P and $\mathcal{U}$

the following holds.

- Let $\tau_1 > \tau_0$. If $d \in \overline{\mathcal{U}(\tau_0, \tau_1)}$ satisfies $P(d; \tau_0, \tau_1) = 0$, $\Psi \in \mathcal{A}(\tau_0, \tau_1, \theta)$ for some $\theta \in (0, 1)$.[Main estimate]

We use the estimate on $\mathcal{A}(\tau_0, \tau_1, \theta)$ and the following form;

$$[e^{-(\tau-\tau_0)A}\phi](y, \tau_0) = \int_0^\infty K(y, \xi, \tau - \tau_0)\phi(\xi, \tau_0)d\xi,$$

Then, the solution ϕ to $\phi_\tau + \mathcal{A}\phi - F(\phi) = 0$ satisfies

$$\phi(y, \tau) = [e^{-(\tau-\tau_0)A}\phi](y, \tau_0) + \int_{\tau_0}^\tau e^{-(\tau-s)A}F(\phi(y, s))ds$$

Properties P and $\mathcal{U}$

Here,

$$K(y,\xi,\theta) = \frac{\exp\left(\frac{N}{4}\theta\right)}{2(1-e^{-\theta})}\frac{\xi^{(N+4)/2}}{y^{(N+2)/2}} \cdot \exp\left(-\frac{e^{-\theta}y^2+\xi^2}{4(1-e^{-\theta})}\right) I_{(N+4\nu+2)/2}\left(\frac{e^{-\theta/2}\xi y}{2(1-e^{-\theta})}\right)$$

Properties P and $\mathcal{U}$

- Let $\tau_1 > \tau_0$. Suppose $\mathcal{U}(\tau_0, \tau_1) \neq \emptyset$, then $\deg(P(\cdot; \tau_0, \tau_1), 0, \mathcal{U}(\tau_0, \tau_1)) = 1$.

In fact, we assume $P(d; \tau_0, \tau) = 0$ with some $d \in \partial\mathcal{U}(\tau_0, \tau_1)$ and $\tau \in [\tau_0, \tau_1]$.
The main estimate leads us to $d \in \mathcal{U}(\tau_0, \tau_1)$. It contradicts.
Then, $P(\partial\mathcal{U}(\tau_0, \tau); \tau_0, \tau) \neq 0$ for $\tau \in [\tau_0, \tau_1]$.
Then,
$\deg(P(\cdot; \tau_0, \tau_1), 0, \mathcal{U}(\tau_0, \tau_1)) = \deg(P(\cdot; \tau_0, \tau_0), 0, \mathcal{U}(\tau_0, \tau_0)) = 1$.

- For $\tau_1 > \tau_0$, $\mathcal{U}(\tau_0, \tau_1) \neq \emptyset$.

Suppose $\tau^* = \sup\{\tau \geq \tau_0 : \mathcal{U}(\tau_0, \tau) \neq 0\} < \infty$.
$\exists d^* \in \overline{\mathcal{U}(\tau_0, \tau^*)}$ such that $P(d^*, \tau_0, \tau^*) = 0$.
Then, the main estimates ensures $d \in \mathcal{U}(\tau_0, \tau, \tau^*)$ and $\mathcal{U}(\tau_0, \tau, \tau^* + \varepsilon) \neq \emptyset$ with some $\varepsilon > 0$. It contradicts.

Proof of $\mathcal{A}(\tau_0, \infty : 1) \neq \emptyset$

Let $\tau_n = \tau_0 + n$. Then, it holds that $\mathcal{U}(\tau_0, \tau_n) \neq \emptyset$ and that $\deg(P(\cdot; \tau_0, \tau_n), 0, \mathcal{U}(\tau_0, \tau_n)) = 1$.
There exists $d_n \in \mathcal{U}(\tau_0, \tau_n)$ satisfying $P(d_n; \tau_0, \tau_n) = 0$.
Then, it holds that $\Psi(\cdot, \cdot; d_n) \in \mathcal{A}(\tau_0, \tau_1; \theta_n)$ with some $\theta_n \in (0, 1)$.
By taking a subsequence, we have $d^* = \lim_{n\to\infty} d_n$.
Then, there exists $\Psi(\cdot, \cdot : d^*) \in \overline{\mathcal{A}(\tau_0, \infty : 1)}$.

Proof of $\mathcal{A}(\tau_0, \infty : 1) \neq \emptyset$

For $\tau \geq \tau_0$, $\Psi = \Psi(\cdot, \cdot; d^*)$ satisfies the followings.

$$\bullet \left| \Psi(y, \tau; d^*) - \Psi_\infty(y) + e^{-\lambda_J \tau} \varphi_J(y) \right| < \theta \varepsilon e^{-\lambda_J \tau} \left(y^{2\nu} + y^{2\lambda_J - 2} \right)$$

for $y \in [K e^{-\eta\tau}, e^{\sigma\tau}]$ and $\tau \in [\tau_0, \tau_1]$,

$$\bullet \Psi_{sub}(\cdot, \tau) < \Psi(\cdot, \tau) < \Psi_{sup}(\cdot, \tau)$$

in $[0, K e^{-\eta\tau}]$,

$$\bullet (1 - 2\varepsilon)\Psi_\infty < \Psi(\cdot, \tau) < (1 + 2\varepsilon)\Psi_\infty$$

in $[e^{\sigma\tau}, \infty)$.

Estimates of u

$\Psi_{sub}(y,\tau) < \Psi(y,\tau) < \Psi_{sup}(y,\tau)$.

$\Psi(y,\tau) = \frac{1}{y^N}\int_0^y z(\xi,\tau)\xi^{N-1}d\xi < \Psi_{sup}(y,\tau)$,
where $z(y,\tau) = (T-t)u(x,t)$.

$\Psi_{sub}(0,\tau)$, $\Psi_{sup}(0,\tau) = O(e^{\lambda_J\tau/(-\nu-1)})$.

$z(0,\tau) = O(e^{\lambda_J\tau/(-\nu-1)})$.

$u(0,t) = O((T-t)^{J/(\nu+1)})$
($J/(\nu+1) \leq -2J/3$ for $J \geq 2$).