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Abstract

Two-dimensional Navier–Stokes equations are solved in an analytical way to clarify characteristics of low-\(Re\) flows in a microscopic channel consisting of two intersecting permeable walls, the intersection of which is supposed to be a sink or a source. Such flows are, therefore, considered to be an extension of the so-called Jeffery–Hamel flow to the permeable wall case. A set of nonlinear forth-order ordinary differential equations are obtained, and their solutions are sought for the small permeable velocity compared with the main flow one by a perturbation method. The solutions contain the solutions found in the past, such as the flow between two parallel permeable walls studied by Berman and the Jeffery–Hamel flow between the impermeable walls as special cases. Velocity distribution and friction loss in pressure along the main stream are represented in the explicit manner and compared with those of the Jeffery–Hamel flow. Numerical examples show that the wall permeability has a great influence on the friction loss. Furthermore, it is shown that the convergent main flow accompanied with the fluid addition through the walls is inversely directed away from the origin due to the balance of the main flow and the permeable one, while the flow accompanied with fluid suction is just directed toward the origin regardless of conditions.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Flows in channels or tubes with permeable walls are seen in not only widespread flows in natural phenomena but also living bodies of humans, animals, insects and plants, etc. One such example in living bodies is flows in blood vessels where vascular walls have permeable properties for supplying tissues with nutrients and clearing waste products (Senger et al. 1983, Michel and Curry 1999, Sundberg et al. 2001, Bates and Harper 2002, Nagy et al. 2008, Kang et al. 2010). The vascular walls therefore allow the free, bidirectional passage of molecules, cells and gases. The study of such flows through vascular permeable walls from the viewpoint of fluid dynamics is therefore vital as basic researches on medical treatments. Although a number of investigations on blood flows have been carried out from the viewpoint of medicine as mentioned above, fluid dynamical characteristics of flows through permeable walls, such as the velocity field and friction loss in pressure on the vascular walls, are not sufficiently investigated probably because of the associated complexities of a fluid-dynamical analysis. It is particularly important for medical researchers to be able to have usable forms of these characteristics, because they are usually unfamiliar with fluid dynamics. In this sense, results based on computational fluid dynamics (CFD) are not so convenient for them, although it is unnecessary to mention that CFD is very important and powerful for study of blood flows.

Pioneering works relating the overall pressure drop to the flow rate in channels with the permeable walls were performed by Olson (1949), Van Der Hegge Zijnen (1951) and Berman (1953). For example, Berman carried out a theoretical analysis on a two-dimensional steady-state laminar flow between parallel permeable walls. He clarified that the velocity profile in the main flow deviates from the Poiseuille parabola by being flatter near the channel center and steeper near the walls depending on Reynolds number on the permeable flow through the channel walls. After the Berman's study, studies on relevant problems have been conducted until now (Mariamma and Majhi 2000, Dinarvand et al. 2009, Si et al. 2016), however, they were not focused on physical characteristics such as the friction loss in pressure, but on rather mathematical aspects such as partial differential equations and numerical procedures. Recent attention to flows in permeable wall tubes is, on the other hand, particularly concerned with flows in a lymphatic drug delivery system (LDDS) that uses the lymphatic network (Kodama et al. 2014, Shao et al. 2015, Kodama et al. 2016). Uses of the LDDS in a precise and reliable way require knowledge of the detailed characteristics of lymphatic flows, especially the pressure loss in lymphatic vessels as the permeable wall tubes.

The most prominent traits of lymphatic vessels are in both their minute size and geometrical irregularity in case of mice Kodama et al. (2014) dealt with. For example, the cross section of a lymphatic vessel varies along its axis like a funnel (Zweifach and Prather 1975, Schmid-Schönbein 1990), and its diameter is of the magnitude of submillimeter (Kodama et al. 2014). Especially, a number of microscopic valves exist in one lymphatic vessel and they repeat contracting or expanding when lymph runs in the vessel. A typical Reynolds number of such a flow is the order of $10^{-2}$ through $10^{-2}$ (Schmid-Schönbein 1990). To understand flows in such microscopic lymphatic vessels, especially valves, as the first step of investigating the flows, Fujikawa et al. (2016) studied a microscopic and small Reynolds number
Jeffery–Hamel flow in a two-dimensional convergent or divergent channel. They derived an extended Bernoulli equation, friction loss and resistance coefficient in exact forms and, consequently clarifying that the friction loss and the resistance coefficient in the finite length of the channel are expressed by explicit functions including both the convergent or divergent angle and Reynolds number; it should be mentioned that the classical Jeffery–Hamel flow is also being revitalized on new problems associated with heat transfer (Turkyilmazoglu 2014, 2015). Following Fujikawa et al’s work (2016) on the Jeffery–Hamel flow, Yaguchi et al (2018) successfully extended it to an axisymmetric case in a tapered tube and confirmed that the increase in the resistance coefficient holds in this case.

It is quite natural that the authors intend to develop the microscopic and small Reynolds number Jeffery–Hamel flow to a flow between two intersecting permeable walls. The aim of this paper is to study a two-dimensional steady-state incompressible laminar flow in a channel consisting of two intersecting permeable walls. The final goal is to understand three-dimensional flows in geometrically irregular lymphatic vessels. The present study is therefore positioned as an extension of the Berman’s analysis of the flow in a permeable parallel channel (Berman 1953) and the Fujikawa et al’s one of the Jeffery–Hamel flow (Fujikawa et al 2016) to flows accompanied with the permeable flow. The velocity distribution of the main flow and the flow normal to it, and the friction loss in pressure along the major flow will be clarified in an analytical way. In section 2, problem statement, governing equations, variable transformation and boundary conditions are presented. In section 3, nonlinear ordinary differential equations describing the flows accompanied with the permeable flow are derived for both convergent and divergent flows, and their perturbation solutions are obtained. In section 4, friction losses defined at different positions in a finite length of convergent channel are derived in convenient forms for practical uses. In sections 5 and 6, flows in both cases of fluid suction and fluid addition through the permeable walls are discussed, and the effects of wall permeability on the main flow field are clarified. Section 7 is the summary of the above findings.

2. Problem statement, governing equations and boundary conditions

We deal with a steady and incompressible flow between two intersecting, semi-infinite, permeable walls shown in figure 1, in which $2\alpha$ is the intersecting angle of the walls. The flow is possible for a divergent or convergent spatial configuration, but in the present study, the convergent flow will be mainly focused on because an instability is expected to take place for the divergent flow (Haines et al 2011) or another reason as will be discussed in section 4.
The cylindrical-polar coordinate system \((r, \phi, z)\) centered on the origin \(O\) is adopted. The flow is symmetric concerning the \(z\)-\(x\) plane and radially directed toward or in the direction away from the origin \(O\). The following assumptions are imposed; (i) the fluid is Newtonian, (ii) no external forces such as gravitation act on the fluid, (iii) the flow is laminar and (iv) the permeable velocity is constant everywhere over the walls. Therefore, the velocity components of the main stream can be expressed as

\[
v_r = v_r(r, \phi), \quad v_\phi = v_\phi(r, \phi), \quad v_z = 0.
\]  

Then, the continuity equation and the two-dimensional Navier–Stokes equations for the main flow are given by Bird \textit{et al} (1960), Landau and Lifshitz (1987)

\[
\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_\phi}{\partial \phi} = 0, \tag{2}
\]

\[
v_r \frac{\partial v_r}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \phi^2} + \frac{1}{r} \frac{\partial v_\phi}{\partial r} - \frac{2}{r^2} \frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2} \right), \tag{3}
\]

\[
v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi} + \nu \left( \frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{2}{r^2} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2} \right), \tag{4}
\]

where \(p\) is the fluid pressure, \(\rho\) is the fluid density and \(\nu\) is the kinematic viscosity of the fluid. The following variable transformation is convenient for the analysis hereafter:

\[
\lambda = \frac{\phi}{\alpha}, \tag{5}
\]

which can be interpreted as the ratio of an arc \(r\phi\) stemming from the \(x\)-axis at an arbitrary distance \(r\) to the arc \(r\alpha\) straddling the \(x\)-axis and the walls. The whole region of the angle \(\phi\) is transformed by (5) into the fixed region of \([-1, 1]\) in the \(\lambda\)-coordinate. We hereafter deal with the main flow by the coordinate system \((r, \lambda)\). It should be here stressed that, for example, in microscopic tubes as lymphatic vessels with the diameter of the magnitude of the viscous sublayer on walls, the fluid runs with a very low Reynolds number (Schmid-Schönbein 1990, Kodama \textit{et al} 2014). Therefore, the flow is laminar.

Now, let us introduce the stream function \(\Psi(r, \lambda)\), by which the velocity components \(v_r\) and \(v_\lambda\) are defined as

\[
v_r = \frac{1}{\alpha r} \frac{\partial \Psi}{\partial \lambda}, \tag{6}
\]

\[
v_\lambda = -\frac{\partial \Psi}{\partial r}. \tag{7}
\]

These velocity components certainly satisfy the continuity equation (2) expressed by cylindrical-polar coordinates (Landau and Lifshitz 1987). As mentioned above, we deal with a small Reynolds number flow in the viscous sublayer, therefore we can sufficiently assume that the angular velocity component \(v_\lambda\) does not depend on the distance \(r\), but on the variable \(\lambda\) alone. This means that \(v_\lambda\) obeys a similarity distribution (Berman 1953). In consequence, integrating (7) with respect to \(r\) leads to

\[
\Psi(r, \lambda) = C(\lambda) - v_\lambda(\lambda)r, \tag{8}
\]
where $C$ is the integral constant and a function of $\lambda$. Substituting (8) into (6) yields

$$v_r(r, \lambda) = \frac{1}{\alpha r} \left( \frac{dC}{d\lambda} - r \frac{d\lambda}{d\lambda} \right)$$

(9)

Therefore, the mass flow rate $Q$ through the arc $2\alpha \lambda$ at the distance $r$ from the origin $O$ can be obtained as

$$Q = 2\alpha \rho \int_0^1 r v_r d\lambda = 2\rho [C(1) - C(0)] - 2\rho r [v_\lambda(1) - v_\lambda(0)],$$

(10)

where the second term of the right-hand side in (10) is the mass flow through the two walls per unit time and unit depth in the direction of $z$. In (10), we put $v_r$ to be

$$v_\lambda(1) = v_w = \text{constant},$$

(11)

$$v_\lambda(0) = 0,$$

(12)

where $v_w$ is the permeable velocity at the walls, in which $v_w > 0$ for the suction from the main flow and $v_w < 0$ for addition into it. Equation (12) is the symmetry condition for $v_\lambda$ at $\lambda = 0$ for the plane-symmetric flow. Concerning the first term in (10), putting $Q(r \to 0) = Q_0$ gives

$$Q_0 = 2\rho [C(1) - C(0)].$$

(13)

The mass flow rate given by (10) can finally be expressed to be

$$Q(r) = Q_0 - 2\rho r v_w.$$  

(14)

Note that $Q < 0$ and $Q_0 < 0$ when the flow through the arc $2\alpha \lambda$ at the distance $r$ is directed toward the origin $O$, while $Q > 0$ and $Q_0 > 0$ when the flow is in the direction away from the origin $O$; in (14), $v_w > 0$ for the permeable velocity of fluid sucked from the main stream into the walls and $v_w < 0$ for the permeable velocity of fluid added into the main stream from the walls. Equation (14) is the universal relation among the three quantities of $Q_0$, $Q_0$ and $r v_w$ on the control surfaces consisting of the origin $O$, the two walls and the arc $2\alpha \lambda$ in the channel. Another modified form will be used for flows in a finite length of channel in section 4.

The forms of (11)–(13) naturally lead us to introduction of the following functions $f(\lambda)$ and $g(\lambda)$, which satisfy the conditions $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$:

$$v_\lambda(\lambda) = v_w f(\lambda),$$

(15)

$$C(\lambda) = \frac{Q_0}{2\rho} g(\lambda).$$

(16)

Therefore, (8) and (9) can respectively be rewritten as

$$\Psi(r, \lambda) = \frac{Q_0}{2\rho} g(\lambda) - v_w r f(\lambda),$$

(17)

$$v_r(r, \lambda) = \frac{1}{\alpha r} \left( \frac{Q_0}{2\rho} \frac{dg(\lambda)}{d\lambda} - v_w f(\lambda) \right).$$

(18)

The velocity components $v_\lambda$ (equation (15)) and $v_r$ (equation (18)), of course, satisfy the continuity equation (2) as they must do. There is the relation $Q = 2\rho \Psi(r, 1)$ between (14) and (17). The second term in the round brackets of the right-hand side of (18) denotes the contribution of the angular velocity $v_\lambda$ to the radial velocity $v_r$, which will bear a term of the order of $v_w^2$ in the second term of the left-hand side of (3) together with the contribution of $v_w$. 


from the third term of it. It should be noted here that these two terms of the order of $v^2_w$ may throw an obstacle to the consistency of a perturbation analysis we will adopt in the following section. Although Berman (1953) assumed the unknown functions as $f(\lambda) = g(\lambda)$, there is no mathematical basis to be able to put it as such form. The two functions are different and should be rigorously treated as different functions. Our problem is now to seek suitable forms of $f(\lambda)$ and $g(\lambda)$ under the following boundary conditions and symmetry conditions at the mid plane:

$$v_r(r, \lambda = \pm 1) = 0,$$

$$v_\lambda(\lambda = \pm 1) = \pm v_w = \text{constant},$$

$$\left(\frac{\partial v_r}{\partial \lambda}\right)_{\lambda=0} = 0,$$

$$v_\lambda(\lambda = 0) = 0,$$

where the double-signs in (20) correspond.

3. Nonlinear ordinary differential equations and perturbation solutions

3.1. Governing equations of $f(\lambda)$ and $g(\lambda)$

Substituting (15) and (18) into (3) and (4) leads to

$$\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{1}{r^3} \frac{Q_0}{2\alpha^2 \rho} \left( \frac{Q_0}{2\rho} g^2 + \frac{\nu}{\alpha^2} g'' \right) - \frac{v_w}{r^2} \left[ \frac{Q_0}{2\alpha^2 \rho} (f'g' + fg'') + \frac{\nu}{\alpha^2} \left( \frac{1}{\alpha^2} f'' + f' \right) \right]$$

$$+ \frac{v^2_w}{r} \left[ f^2 + \frac{1}{\alpha^2} g'' \right] - \frac{1}{\alpha^2} \frac{\partial p}{\partial \lambda} = \frac{1}{r^2} \frac{Q_0}{2\alpha^2 \rho} g'' - \frac{v_w}{r} \left[ \frac{Q_0}{2\rho} f' + \frac{\nu}{\alpha^2} (f'' + cf) \right],$$

where the prime denotes differentiation with respect to $\lambda$. The dependence of the pressure $p$ on the variables $r$ and $\lambda$ in (23) and (24) is completely different from those of the corresponding equations for the Berman flow (1953) in the point that the derivatives $\partial p/\partial r$ and $\partial p/\partial \lambda$ are dependent on the variables $\lambda$ and $r$ in (23) and (24), respectively. Scrutinizing all terms with $v_w$ in (23) and (24), we notice the following points:

(i) the permeable velocity appears up to the order of $v^2_w$ in (23),

(ii) it appears up to the order of $v_w$ in (24) where the two terms of the order of $v^2_w$, i.e., the second term of the left-hand side of (4) and the third term stemming from the product of the second term of $v_r$ in (18) by $v_w$, are canceled out.

Considering that (23) and (24) will be solved by means of a perturbation method in the following, we should first make the highest terms on $v_w$ in (23) and (24) of the same order ones. If the third term of the right-hand side of (23) is negligibly small compared with the other ones, it can be ostracized as discussed in appendix A at the end of this paper. The reasonableness of such treatment of (23) will be justified in a later stage by showing that (23) can bear the well-known solution of Jeffery–Hamel flow (Esmaeilpour and Ganji 2010, Fujikawa et al 2016) and (24) can also yield the solution of Berman flow (1953). Then we
must note that solutions of \( f(\lambda) \) and \( g(\lambda) \) are correct up to the order of at most \( v_w \). Therefore, the equation (23) can reasonably be approximated to be

\[
\frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{1}{r^3} \frac{Q_0}{2\alpha^2 \rho} \left( \frac{Q_0}{2\rho} \left( \frac{2}{\alpha^2} f^{(4)} g'' + f'' g'' + f'' f'' \right) + \frac{v_w}{r^2} \left( \frac{1}{\alpha^2} f''' + f' \right) \right).
\]  \tag{25}

Eliminating the pressure in (24) and (25), we get the following equation:

\[
\frac{1}{r^3} \left[ \frac{1}{2} \left( \frac{Q_0}{\alpha \rho} \right)^2 g'''' + \frac{\nu Q_0}{\alpha \rho} \left( \frac{1}{\alpha^2} g^{(4)} + 2g'' \right) \right] - \frac{1}{r^2} \frac{v_w Q_0}{2\rho} \left( \frac{1}{\alpha^2} (2f'' g'' + f'' g'' + f'' f'') \right) + f' + \nu \omega \left( \frac{1}{\alpha^2} f^{(4)} + \frac{2}{\alpha} f'' + \omega f' \right) = 0,
\]  \tag{26}

where \( f^{(4)} = d^4 f(\lambda)/d\lambda^4 \) and \( g^{(4)} = d^4 g(\lambda)/d\lambda^4 \). The insides of the two square brackets must be zero because (26) always holds for an arbitrary \( r \) under \( Q_0/(\alpha \rho) \equiv 0 \) and \( v_w \equiv 0 \). Then we obtain the following two equations among \( f, g \) and their derivatives:

\[
\frac{Q_0}{2\alpha \rho} g'''' + \nu \left( \frac{1}{\alpha^2} g^{(4)} + 2g'' \right) = 0,
\]  \tag{27}

\[
\frac{Q_0}{2\rho} \left( \frac{1}{\alpha^2} (2f'' g'' + f'' g'' + f'' f'') \right) + \nu \left( \frac{1}{\alpha^2} f^{(4)} + \frac{2}{\alpha} f'' + \omega f' \right) = 0.
\]  \tag{28}

In case where the walls are impermeable, i.e., \( v_w = 0 \), (27) alone remains and (28) is irrelevant. Equation (27) is composed of only derivatives of \( g(\lambda) \). Therefore, the function \( g(\lambda) \) which is the main part describing the radial velocity component \( v_r \) (equation (18)) should get identical with the solution of the Jeffery–Hamel flow for \( v_w = 0 \) (Landau and Lifshitz 1987, Esmaeilpour and Ganji 2010, Fujikawa et al 2016) as will be shown in section 3.2. Equation (28) is the equation governing the angular velocity component \( v_\omega \). The quantity \( |Q_0|/(\rho \nu) \) which can be defined from (27) and (28) is the Reynolds number in the Jeffery–Hamel flow (Landau and Lifshitz 1987, Fujikawa et al 2016). Therefore, we define the following Reynolds number:

\[
Re = \frac{|Q_0|}{\rho \nu},
\]  \tag{29}

where the numerator of the right-hand side in (29) is the mass flow rate of the sink or the source positioned at the origin \( O \). The flow around the origin \( O \) is directed toward the origin \( O \) for \( Q_0 < 0 \), on the other hand, the flow around the origin \( O \) is directed in the direction away from the origin \( O \) for \( Q_0 > 0 \). Equations (27) and (28) are the nonlinear ordinary differential equations describing the divergent or convergent flow between the two intersecting permeable walls.

### 3.2. Perturbation solutions of velocities

In this subsection, we will focus on the flow between the two intersecting permeable walls where the main flow is convergent in the direction of the origin \( O \), i.e., \( Q_0 < 0 \) and the magnitude of its velocity is much larger than that of the permeable flow. The former is because instability is expected to take place in the divergent flow (Haines et al 2011) and also the fluid supplied from the origin with a finite flow rate cannot run beyond a certain distance, as will be shown in a later stage. Expressing \( Q_0 < 0 \) as \( Q_0 = -|Q_0| = -\rho \nu Re \) by (29), we
can rewrite (27) and (28) to be

\[ \text{Re} \alpha g'^n - g'^{(4)} - 4\alpha^2 g'' = 0, \]

(30)

\[ \text{Re} \{2f'^n + f''g' + fg''\} + \alpha^3 f'^{\alpha} - 2(f'^{(4)} + 2\alpha^2 f'' + \alpha^4 f) = 0. \]

(31)

Equation (30) shows that \( g'^n \)'s are not coupled with \( f'^\alpha \)'s, which means that we can get the solution of this equation independently of \( f'^\alpha \)'s. Once \( g'^n \)'s are known, we can solve (31) in the following way. Let us expand the functions \( g(\lambda) \) and \( f(\lambda) \) in the forms of series expansions by the angle \( \alpha \) as follows:

\[ g(\lambda) = g_0(\lambda) + g_1(\lambda)\alpha + g_2(\lambda)\alpha^2 + \cdots + g_n(\lambda)\alpha^n + \cdots, \]

(32)

\[ f(\lambda) = f_0(\lambda) + f_1(\lambda)\alpha + f_2(\lambda)\alpha^2 + \cdots + f_n(\lambda)\alpha^n + \cdots, \]

(33)

where the \( g_n \)'s and \( f_n \)'s are functions to be sought. Substituting (32) into (30) and sorting all terms according to powers of \( \alpha \) up to its second order lead to the following set of equations;

Zeroth order: \( g_0' = 0 \),

First order: \( g_1'^{(4)} - \text{Re} \{g_0'g_1''\} = 0 \),

Second order: \( g_2'^{(4)} + 4g_0'' - \text{Re} \{g_0'g_1'' + g_0''g_1'\} = 0 \).

Furthermore, substituting (32) and (33) into (31) and sorting terms according to powers of \( \alpha \) lead to the following set of equations;

Zeroth order: \( f_0'^{(4)} = 0 \),

First order: \( 2f_1'^{(4)} - \text{Re} \{2f_0'g_1'' + f_0''g_1' + f_0''g_0''\} = 0 \),

Second order: \( 2f_2'^{(4)} + 4g_0'' - \text{Re} \{f_0'g_1'' + f_1''g_0'' + 2f_0'g_1'' + f_0''g_1' + f_1''g_1'\} = 0 \).

The boundary conditions for the \( g_n \)'s and \( f_n \)'s can be obtained from (19)–(22). As the boundary conditions of \( g(\lambda) \) are \( g(0) = 0 \) and \( g(1) = 1 \), the \( g_n \)'s can be determined to be

\[ g_n(0) = 0, \ (n = 0, 1, 2, 3, \ldots), \]

(40)

\[ g_n(1) = 1, \]

\[ g_n(1) = 0, \ (\text{for } n \geq 1). \]

(41)

From (18), (19) and (21), we furthermore obtain

\[ g'(\pm 1) = -\frac{2\nu_f}{\nu Re} f'(\pm 1), \]

(42)

\[ g''(0) = -\frac{2\nu_f f''(0)}{\nu Re}. \]

(43)

The above two constraints should hold for arbitrary \( r \) and \( \nu_f \), so that \( g'(\pm 1) = f'(\pm 1) = 0 \) and \( g''(0) = f''(0) = 0 \) must be satisfied. Thus, we have

\[ g_n'(\pm 1) = 0, \ (n = 0, 1, 2, 3, \ldots), \]

(44)

\[ f_n'(\pm 1) = 0, \ (n = 0, 1, 2, 3, \ldots). \]

(45)
\[ g_n''(0) = 0, \quad (n = 0, 1, 2, 3, \ldots), \quad (46) \]
\[ f_n''(0) = 0, \quad (n = 0, 1, 2, 3, \ldots). \quad (47) \]

From (15), (20) and (22), we obtain
\[ f_n(0) = 0, \quad (n = 0, 1, 2, 3, \ldots), \quad (48) \]
\[ f_0(\pm 1) = \pm 1, \]
\[ f_n(\pm 1) = 0, \quad (\text{for } n \geq 1), \quad (49) \]

from which \( f(0) = 0 \) and \( f(1) = 1 \) are satisfied. All boundary conditions necessary for determining the \( g_n' \)'s and \( f_n' \)'s have been given in the above.

We will seek out the zeroth-order solutions \( g_0(\lambda) \) and \( f_0(\lambda) \) of the unknown functions \( g(\lambda) \) and \( f(\lambda) \). By direct integration of (34) under its boundary conditions (40), (41), (44) and (46), we can first find
\[ g_0(\lambda) = \frac{1}{2} \lambda(3 - \lambda^2). \quad (50) \]

Similarly, by direct integration of (37) under its boundary conditions (45), (47)–(49), we can obtain
\[ f_0(\lambda) = \frac{1}{2} \lambda(3 - \lambda^2). \quad (51) \]

Consequently, from (15), (17) and (18), the zeroth-order stream function \( \Psi_0(r, \lambda) \), and the zeroth-order velocity components \( v_{r0}(r, \lambda) \) and \( v_{\theta0}(\lambda) \) are respectively determined to be
\[ \Psi_0(r, \lambda) = -\frac{1}{4}(\nu Re + 2v_\theta r)\lambda(3 - \lambda^2), \quad (52) \]
\[ v_{r0}(r, \lambda) = -\frac{3}{4\alpha r}(\nu Re + 2v_\theta r)(1 - \lambda^2), \quad (53) \]
\[ v_{\theta0}(\lambda) = \frac{1}{2}v_\theta \lambda(3 - \lambda^2), \quad (54) \]

where the subscript ‘0’ in \( \Psi_0, v_{r0} \) and \( v_{\theta0} \) denotes the zeroth-order quantities. In the above, all zeroth-order terms have been determined. Equations (52)–(54) respectively become exactly identical with the zeroth-order expressions of the Berman flow (1953) by putting \( Re = |Q_0|/\nu = -2h\bar{\pi}_{\theta0}(0)/\nu \), where \( \bar{\pi}_{\theta0}(0) \) is the average velocity at \( x = 0 \), and also by \( h/(\alpha r) \to 1 \) for \( r \to \infty \):
\[ \Psi_0^B(r, \lambda) = \frac{1}{2}[h\bar{\pi}_{\theta0}(0) - v_\theta r]\lambda(3 - \lambda^2), \quad (55) \]
\[ v_{r0}^B(x, \lambda) = \frac{3}{2}[\bar{\pi}_{\theta0}(0) - \frac{v_\theta x}{h}](1 - \lambda^2), \quad (56) \]
\[ v_{\theta0}^B(\lambda) = \frac{1}{2}v_\theta \lambda(3 - \lambda^2), \quad (57) \]

where the radial distance \( r \) is replaced by \( x \) and superscript ‘\( B \)’ denotes the Berman flow. Equations (55) and (56) hold under the condition, \( 0 \leq x/h \leq \bar{\pi}_{\theta0}(0)/v_\theta \) for \( \bar{\pi}_{\theta0}(0) > 0 \) and \( v_\theta > 0 \), as clarified by the Berman’s paper (1953). Putting \( v_\theta = 0 \) in (53) and (54), on the other hand, they automatically reduce to the leading terms of velocity components of the...

\[ \nu_{\text{HI}}^0(r, \lambda) = -\frac{3\nu Re}{4\alpha r}(1 - \lambda^2), \]  

\[ \nu_{\text{HI}}^0(\lambda) = 0, \]  

where \( Q_0 = Q \) from (14) for \( v_w = 0 \). That is, our present solutions given by (52)–(54) are found to include the previous zeroth-order solutions for the flow between two-dimensional parallel, permeable walls and also the well-known solutions of the Jeffery–Hamel flow. These facts surely provide the present solutions with the evidence of the appropriateness for the perturbation analysis.

Hereafter, the first-order solutions for the convergent flow will be sought out. From (35) and its boundary conditions (40), (41), (44) and (46), we can determine the function \( g_1(\lambda) \) independent of \( f(\lambda) \) as

\[ g_1(\lambda) = -\frac{3}{560} Re \lambda(1 - \lambda^2)^2(5 - \lambda^2). \]  

Furthermore, using \( g_0 \) and \( f_0 \) given by (50) and (51) enables us to get the following equation from (38):

\[ f_1^{(4)} = -\frac{3}{2} Re \lambda(6 - 5\lambda^2). \]  

By direct integration of (61) under the \( f_1 \)'s boundary conditions (45), (47)–(49), the function \( f_1(\lambda) \) can be determined to be

\[ f_1(\lambda) = -\frac{1}{560} Re \lambda(1 - \lambda^2)^2(32 - 5\lambda^2). \]  

In summarizing the above results, we determined the unknown functions \( g(\lambda) \) and \( f(\lambda) \), the stream function \( \Psi(r, \lambda) \) (equation (17) ) and the velocity components \( v_r \) (equation (18)) and \( v_\lambda \) (equation (15)) up to the order of \( \alpha^2 \) to be

\[ g(\lambda) = \frac{1}{2} \lambda(3 - \lambda^2) - \frac{3}{560} \alpha Re \lambda(1 - \lambda^2)^2(5 - \lambda^2), \]  

\[ f(\lambda) = \frac{1}{2} \lambda(3 - \lambda^2) - \frac{1}{560} \alpha Re \lambda(1 - \lambda^2)^2(32 - 5\lambda^2), \]  

\[ \Psi(r, \lambda) = -\frac{1}{4} (\nu Re + 2v_w r) \lambda(3 - \lambda^2) + \frac{1}{1120} \alpha Re \lambda(1 - \lambda^2)^2 \times [3\nu Re(5 - \lambda^2) + 2v_w r(32 - 5\lambda^2)], \]  

\[ v_r(r, \lambda) = -\frac{3}{4\alpha r}(\nu Re + 2v_w r)(1 - \lambda^2) + \frac{1}{1120 r} Re(1 - \lambda^2)[3\nu Re(5 - 28\lambda^2 + 7\lambda^4) + 2v_w r(32 - 175\lambda^2 + 35\lambda^4)], \]  

\[ v_\lambda(\lambda) = \frac{1}{2} v_w \lambda(3 - \lambda^2) - \frac{1}{560} \alpha Re v_w \lambda(1 - \lambda^2)^2(32 - 5\lambda^2). \]  

In the above, all first-order correction terms have been determined. Note that \( \Psi(r, \lambda), v_r, \) and \( v_\lambda \) are the sums of the zeroth-order solution and the first-order one of \( \Psi(r, \lambda), v_r, \) and \( v_\lambda, \) respectively. As discussed in the beginning of section 2, \( v_\lambda \) is the function of the variable \( \lambda \)
alone, but it includes the effect of the main flow through the Reynolds number Re. The leading term of $v_{l1}$, (67), is expressed by the cubic function of $\lambda$ and its correction term of the 7th-order function of $\lambda$, and the effect of Re on $v_{l1}$ appears in the correction term which is smaller than the leading term by the order of one-thousand for Re = 1.

The second-order correction terms are similarly obtainable through a laborious derivation process. They are given and compared graphically with the zeroth-order terms and the first-order corrections ones in appendix B. Figure B1 demonstrates that the first- and second-order correction terms in both $g(\lambda)$ and $f(\lambda)$, i.e., $|g_0\alpha|$, $|g_2\alpha^2|$, $|f_1\alpha|$ and $|f_2\alpha^2|$ are respectively much smaller than the zeroth-order terms, i.e., $|g_0|$ and $|f_0|$, for Re = 1 and $\alpha = 32^\circ$. The angle of $\alpha = 32^\circ$ is the maximum value used for the calculation in section 5. This figure clearly shows that the zeroth-order terms are predominant in series expansions of $g(\lambda)$ and $f(\lambda)$, and that the expansions are valid for $\alpha = 32^\circ$; it is numerically ascertained that the expansions converge for even an angle larger than this value, if the angle is less than unity, and for the larger Reynolds number, e.g., Re = 10. Further higher-order correction terms are obtainable, but such expansion is un-necessary.

Well, although the zeroth-order solutions of $g(\lambda)$ and $f(\lambda)$ are equal, they are different in the higher order approximation as they are expressed by (63) and (64). The assumption, $g(\lambda) = f(\lambda)$, put by Berman (1953) for the parallel walls breaks down from the first-order correction for the intersecting walls, although its effect on the accuracy of the solution is relatively small. The above results are relevant for the convergent flow, i.e., $Q_0 < 0$. However, a comment on the divergent flow, i.e., the flow of $Q_0 > 0$, should briefly be given here. We can obtain the solutions for the divergent flow by replacing Re in the solutions for the convergent flow with $-Re$ or $-|Q_0|/(\rho \nu)$, and the zeroth-order solutions of $g(\lambda)$ and $f(\lambda)$ become the same ones as (50) and (51). From (53), the velocity component $v_{l0}(r, \lambda)$ takes the form

$$v_{l0}(r, \lambda) = -\frac{3}{4\alpha r} \left( \frac{|Q_0|}{\rho} + 2\nu_{w} r \right) (1 - \lambda^2).$$  

Equation (68) suggests that there exists the point where the fluid cannot run beyond it for the divergent flow and the fluid suction, i.e., $\nu_{w} > 0$. We can find the point where the fluid exists by putting $Q > 0$ for the fluid suction from the main stream through the permeable walls, i.e., $\nu_{w} > 0$ in (14) to be

$$r < \frac{|Q_0|}{2\rho \nu_{w}}.$$  

We can ascertain, by (68), that $v_{l0} = 0$ for every $\lambda$ at $r = |Q_0|/(2\rho \nu_{w})$ as it must do. However, in the following we will only focus on the convergent channel where the origin is a sink because, for the flow in the divergent channel, the fluid disappears somewhere downstream as discussed above and instability may take place (Stow et al 2001, Haines et al 2011).

4. Friction loss in finite length of convergent channel

The most interesting phenomenon in the convergent flow between the two intersecting permeable walls is the pressure loss in the channel, on which Fujikawa et al (2016) have found the drastic increase in pressure loss due to viscosity for the Jeffery–Hamel flow. This conclusion was recently found to hold for the axisymmetric flow in the tapered tube by Yaguchi et al (2018). We will here look at the effect of permeability on the pressure loss for the permeable flow. The pressure distribution $p(r, \lambda)$ in the convergent channel is readily
obtained from (24) or (25). Putting $Q_0 = -|Q_0|$ for the convergent flow and integrating (25) with respect to $r$, and we obtain

$$p(r, \lambda) + \frac{|Q_0|^2}{8\alpha^2 r^2} r'' + \frac{\nu_0 |Q_0|}{2\alpha^2 r} (f'g' + fg'') + H(r, \lambda) = p(\infty, 0),$$  

(70)

where an integral constant has been determined at $r \to \infty$ and $\lambda = 0$ to be $p(\infty, 0)$ and the function $H(r, \lambda)$ is given by

$$H(r, \lambda) = -\frac{\nu |Q_0|}{4\alpha^2 r^2} g'' - \frac{\nu_0}{\alpha^3 r} (f'' + \alpha^2 f').$$  

(71)

Equation (70) is the pressure equation, in which the second- and third-terms of the left-hand side are kinetic energy terms and the function $H(r, \lambda)$ is the friction loss in pressure unit. Here, we adopt the zeroth-order solutions $g_0 = f_0 = \lambda(3 - \lambda^2)/2$, i.e., (50) and (51) as $g(\lambda)$ and $f(\lambda)$ because the higher-order quantities are very small compared with the zeroth-order ones. When we express $H(r, \lambda)$ for the zeroth-order solution of $H(r, \lambda)$ to be $H_0(r, \lambda)$, we have

$$H_0(r, \lambda) = \frac{3\nu |Q_0|}{4\alpha^3 r^2} + \frac{3\nu_0}{\alpha^3 r} \left(1 - \frac{1 - \lambda^2}{2\alpha^2}\right),$$  

(72)

where the first term of the right-hand side of (72) is the one for the Jeffery–Hamel flow and the second term is the effect of permeable flow through the walls. For example, for $\nu_0 = 0$, $H_0$ reduces to

$$H_0^{\text{HI}}(r) = \frac{3\nu |Q_0|}{4\alpha^3 r^2},$$  

(73)

where $H_0^{\text{HI}}$ is the leading term of the friction loss for the Jeffery–Hamel flow (Fujikawa et al. 2016); note that the friction loss in their work is given up to the higher-order term of $O((\text{Re}/\alpha)^2)$. On applying (72) to a flow in the finite length of the channel, $Q_0$ will be replaced by another expression keeping the correctness of this equation. For this purpose, let us return back to (14) and change a little its form as follows; (14) is the universal relation among the three quantities of $Q(r), Q_0$ and $2\rho \nu_0 w$ on the control surfaces consisting of the origin $O$, the two walls and the arc $2r\alpha$ in the channel. A convenient form modified without losing the correctness may be

$$Q(r) = Q(r_1) + 2\rho r_1 \nu_0 w - 2\rho \nu w.$$  

(74)

Comparing (14) and (74), we notice that $Q_0$ in (14) is replaced by the sum of the first and second terms of the right-hand side in (74), i.e., $Q_0 = Q(r_1) + 2\rho r_1 \nu_0 w$. This is another expression of the mass conservation in the channel. Therefore, we replace $Q_0$ by $Q(r_1) + 2\rho \nu_0 w$. This is because the finite length of channel does not contain the origin $O$, but exists between the surfaces at the distances at $r = r_1$ and $r = r_2$, respectively. An important and useful form of (72) is the friction loss $\Delta H$ defined in a finite region of channel, which is practically useful. Applying (72) to the finite region from the cross section 1 at $r = r_1$ to the cross section 2 at $r = r_2$ as shown in figure 2 leads to the following expression:

$$\Delta H = H_{02} - H_{01} = \frac{3\nu |Q(r_1)| + 2\rho r_1 \nu_0 w}{4\alpha^3} \left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) + \frac{3\nu_0}{\alpha^3} \left(1 - \frac{1 - \lambda^2}{2\alpha^2}\right) \left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right).$$  

(75)
The ratio of $D_H$ to $\Delta H^{\text{II}} = \Pi_0^{\text{II}}(r_2) - \Pi_0^{\text{II}}(r_1)$ is expressed as
\[
\frac{\Delta H}{\Delta H^{\text{II}}} = 1 + \frac{2Re_w}{Re} \left( 1 - \frac{1 - \frac{\lambda^2}{2}}{2} \right),
\] (76)
where $Re_w$ is Reynolds number for the permeable flow, which is defined by the permeable velocity $v_w$ and the harmonic mean of two radial positions, i.e., $r_1$ and $r_2$, as follows:
\[
Re_w = \frac{|v_w|}{\nu} \cdot \frac{2r_1 r_2}{r_1 + r_2},
\] (77)
in which the factor $2r_1 r_2/(r_1 + r_2)$ is the harmonic mean of the radii $r_1$ and $r_2$. The ratio of $Re_w$ to $Re$ in (76) can be regarded as the parameter that denotes the effect of the permeable flow through the walls on the main stream flow. Therefore, the ratio of the friction loss $\Delta H_c$ along the axis ($\lambda = 0$) to $\Delta H^{\text{II}}$ is represented by
\[
\frac{\Delta H_c}{\Delta H^{\text{II}}} = 1 + \frac{2Re_w}{Re} \left( 1 - \frac{1}{2} \frac{\alpha^2}{\lambda^2} \right),
\] (78)
on the other hand, the ratio of the friction loss $\Delta H_w$ along the permeable walls ($\lambda = \pm 1$) to $\Delta H^{\text{II}}$
\[
\frac{\Delta H_w}{\Delta H^{\text{II}}} = 1 + \frac{2Re_w}{Re}.
\] (79)

Although the friction loss $\Delta H$ given by (75) depends on the similarity variable $\lambda$ and the wall positions of the finite region, i.e., $r = r_1$ and $r = r_2$, the cross sectional average $\Delta H$ is independent of $\lambda$ as follows:
\[
\Delta H = \overline{H}_{02} - \overline{H}_{01} = \frac{3\nu Q(r_1)}{4\alpha^3} \left( \frac{1}{r_2^2} - \frac{1}{r_1^2} \right) + \frac{3\nu v_w}{\alpha^3} \left( 1 - \frac{1}{3} \frac{\alpha^2}{\lambda^2} \right) \left( \frac{1}{r_2} - \frac{1}{r_1} \right),
\] (80)

The ratio of $\Delta H$ to $\Delta H^{\text{II}}$ is finally given by
\[
\frac{\Delta H}{\Delta H^{\text{II}}} = 1 + \frac{2Re_w}{Re} \left( 1 - \frac{1}{3} \frac{\alpha^2}{\lambda^2} \right).
\] (81)

It is found from (81) that the effect of flow through the permeable walls become linearly larger with the inclination of $2(1 - \alpha^2/3)$ according to the increase in $Re_w/Re$ for the convergent flow.
5. Flows with fluid suction through permeable walls

Here we quantitatively show velocity distributions, streamlines and friction loss in the finite region of the convergent channel where the main stream flow is sucked into the walls. The channel dimensions are the same as those in Fujikawa et al’s paper (2016) and Yaguchi et al’s one (2018): \( \alpha = 16^\circ \), \( r_1 = 0.32 \) mm, \( r_2 = 0.16 \) mm, thus \( L = r_1 - r_2 = 0.16 \) mm and \( h_1 = 2r_1 \alpha = 0.179 \) mm as shown in figure 3(a). These dimensions are given by visualization of actual lymphatic vessel of mouse. We consider the situation where the convergent angle \( \alpha \) varies from \( \alpha = 16^\circ \) (figure 3(a)) to \( \alpha = 32^\circ \) (figure 3(c)) keeping both \( h_1 = 0.179 \) mm and \( L = 0.16 \) mm. Figure 3(b) shows the configuration between figures 3(a) and (c). Both \( r_1 \) and \( r_2 \) accordingly change owing to the shift of the origin when \( \alpha \) varies. The open ratio of the convergent channel, \( \beta \), is defined as the ratio of cross section area at the exit, i.e., at \( r = r_2 \) to the one at the angle \( \alpha = 16^\circ \) where \( \beta = 1 \). The open ratio for \( \alpha = 32^\circ \) is \( \beta = 0 \). The fluid is water: the density and kinematic viscosity are respectively \( \rho = 1000 \) kg m\(^{-3}\) and \( \nu = 1.0 \times 10^{-6} \) m\(^2\) s\(^{-1}\).

Figure 4 shows streamlines, velocity vector diagrams and pressure distributions for both (a) \( Re = 0.1 \) \( (Q_0 = Q(r_1) + 2pr_1v_w = -0.1 \) g s\(^{-1}\)\) and (b) \( Re = 1 \) \( (Q_0 = Q(r_1) + 2pr_1v_w = -1 \) g s\(^{-1}\)\) in the configuration figure 3(a); the streamlines are represented by thin black solid lines, the velocity vectors by red arrows and the pressure distributions by colored stripes. The thick line in each figure is the upper wall. Note that \( Re \) is Reynolds number of the main flow. The permeable velocities are \( v_w = 0.1 \) mm s\(^{-1}\) for both cases of \( Re = 0.1 \) and 1.

For comparison, the impermeable wall cases \( (v_w = 0) \) are correspondingly represented in the left column in figure 4. For \( \alpha = 16^\circ (\beta = 1) \), the friction loss for the permeable case is more than 1.4 times larger than the Jeffery–Hamel flow as will lately be shown in figure 7. Due to the symmetricity of the flows for both cases, only the upper flow domain is indicated in all figures. It is found that the streamlines of the main flows for both permeable and impermeable cases are different especially in the neighborhood of the wall, i.e., the main flow of the permeable case is prominently directed normal to the wall. Referring to the validity of the approximated equation (25) instead of (23), we have the condition of \( \left( \frac{v_w}{U} \right)^2 = 0.046 \ll 1 \) for \( Re = 0.1 \) and \( \left( \frac{v_w}{U} \right)^2 = 4.551 \times 10^{-4} \) for \( Re = 1 \) when \( v_w = 0.1 \) mm s\(^{-1}\); here \( |U| \) is the reference velocity defined in appendix A. Therefore, the equation (25) holds sufficiently.

Figure 5 shows the relation between the shear stress \( \tau_w \) on the wall and the angle \( \alpha \) for \( Re = 0.1 \) (left) and \( Re = 1 \) (right) at the midpoint of the channel, i.e., \( r = (r_1 + r_2)/2 \). The abscissa denotes the angle \( \alpha \), while the ordinate does the shear stress \( \tau_w \). The permeable
Figure 4. Streamlines, velocity vector diagrams and pressure distributions of the flows for (a) $Re = 0.1$ and (b) $Re = 1$. 
velocities are $v_w = 0.1 \text{ mm s}^{-1}$ for both cases of $Re = 0.1$ and 1. The shear stress is given by

$$\tau_w = \rho \mu \left[ \frac{3\nu Re}{2\alpha^2 r^2} \left( 1 + \frac{2}{35} \alpha Re \right) + \frac{3v_w}{\alpha^2 r} \left( 1 + \frac{9}{70} \alpha Re \right) \right]. \quad (82)$$

where the first approximation of radial velocity, i.e., the equation (66) is used. The flow rate $|Q|$ varies along the axis depending on both the angle and the suction velocity for the permeable wall, e.g., the flow rate at the midpoint of the channel in figure 5 is 0.12–0.15 g s$^{-1}$ for $Re = 0.1$ and 1.02–1.05 g s$^{-1}$ for $Re = 1$, while 0.10 g s$^{-1}$ for $Re = 0.1$ and 1.00 g s$^{-1}$ for $Re = 1$ for the impermeable wall, therefore when the flow rate is shown for the permeable wall, the position where it is measured and the angle of the channel must be designated. For comparison, the impermeable case is shown by the broken lines. It is found that the shear stress for the permeable case is larger than that for the impermeable one. For example, in case of $Re = 1$, the larger the convergent angle is, the largely the shear stresses change for both permeable and impermeable cases, while the shear stresses little change for $Re = 0.1$.

Figure 6 shows the friction loss $\Delta \Pi$ given by (80) against $\alpha$ for $Re = 0.1$ (left) and $Re = 1$ (right). The permeable velocities are $v_w = 0.1 \text{ mm s}^{-1}$ for both cases of $Re = 0.1$ and 1. The

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**Figure 5.** Shear stress on the wall at $r = (r_1 + r_2)/2$ for $Re = 0.1$ and $Re = 1$.

**Figure 6.** Friction loss $\Delta \Pi$ of the finite channel for $Re = 0.1$ and $Re = 1$.
friction loss becomes drastically large depending on Reynolds number of the main flow as \( \alpha \) approaches \( \alpha = 32^\circ \) (open ratio \( \beta = 0 \)); the larger the value of \( Re \) is, the more prominent the increase tendency of \( D_{HH} \) becomes.

Figure 7 compares the friction loss ratios \( D_{HH}/D_{HH}^{\text{ave}} \) (red), \( \Delta H_{w}/\Delta H_{w}^{\text{ave}} \) (blue) and \( \Delta H_{w}/\Delta H_{w}^{\text{ave}} \) (purple) against the convergent angle for \( Re = 0.1 \) (left) and \( Re = 1 \) (right). The permeable velocities are \( v_{w} = 0.1 \text{ mm s}^{-1} \) for both cases of \( Re = 0.1 \) and 1. All the loss ratios approach unity as \( \alpha \) reaches \( \alpha = 32^\circ \) (open ratio \( \beta = 0 \)). This means that the discrepancy in the pressure losses of the permeable and impermeable cases becomes identical near the close-up angle of \( \alpha = 32^\circ \). All the loss ratios, on the other hand, become large as \( \alpha \) approaches \( \alpha = 16^\circ \) (\( \beta = 1 \)). For example, for the flow with \( Re = 0.1 \), the friction loss of the permeable flow is more than 1.4 times compared with the impermeable case. The discrepancy at the smaller angle becomes large according to the decrease in \( Re \) as we can see from the decrease of the second term in the right side of (78), (79) and (81).

6. Flows with fluid addition through permeable walls

In this final section, we deal with the convergent flow of \( Q_{0} < 0 \) with fluid addition, i.e., \( v_{w} < 0 \) into the main stream through the permeable walls. Figure 8 shows streamlines, velocity vectors, and pressure distributions of the flow with \( v_{w} = -0.1 \text{ mm s}^{-1} \) (left column) and \( v_{w} = -0.2 \text{ mm s}^{-1} \) (right column) for both (a) \( Re = 0.1 \) (\( Q_{0} = Q(r_{1}) + 2 p r_{1} v_{w} = -0.1 \text{ g s}^{-1} \)) and (b) \( Re = 1 \) (\( Q_{0} = Q(r_{1}) + 2 p r_{1} v_{w} = -1 \text{ g s}^{-1} \)) in the configuration figure 3(a); the streamlines are represented by thin black solid lines, the velocity vectors by red arrows and the pressure distributions by colored stripes. It is noticeable that, as shown in figure 8(b), the main flow with the larger Reynolds number (\( Re = 1 \)) becomes the convergent flow directed toward the origin on the whole, while, as shown in figure 8(a), the main flow with the smaller Reynolds number (\( Re = 0.1 \)) for \( v_{w} = -0.2 \text{ mm s}^{-1} \) is directed toward the origin in the narrower half region and in the direction away from the origin in the wider half region. In the right-hand side of figure 8(a), the reverse flow occurs because the permeable flow velocity is relatively larger than the velocity of main flow directed toward the origin in the wider half region.
Figure 8. Streamlines, velocity vector diagrams and pressure distributions of the flow with permeable velocities $v_w = -0.1$ mm s$^{-1}$ and $v_w = -0.2$ mm s$^{-1}$ for (a) $Re = 0.1$ and (b) $Re = 1$. 
Then, it is interesting to ask what condition of the permeable velocity the inverse flow in the direction away from the origin occurs at? Let us look at the velocity distribution along the symmetrical axis. The first approximation of \( v_r \), i.e., (66) for \( \lambda = 0 \) is

\[
v_r(0, 0) = -\frac{3\nu Re}{4\alpha r} \left( 1 - \frac{1}{56} \alpha Re \right) - \frac{3v_w}{2\alpha} \left( 1 - \frac{4}{105} \alpha Re \right),
\]

where \( \alpha Re < 105/4 \). The second term of the right-hand side of (83) is the asymptote in case of \( r \to \infty \), i.e.,

\[
v_r(\infty, 0) \to -\frac{3v_w}{2\alpha} \left( 1 - \frac{4}{105} \alpha Re \right),
\]

and \( v_r(0, 0) \to -\infty \). This means that the asymptote \( v_r(r \to \infty, 0) \) becomes negative for \( v_w > 0 \), i.e., the fluid in the main stream is sucked through the walls, and then the velocity distribution along the symmetrical axis becomes negative for an arbitrary position \( r \); the flow is directed toward the origin everywhere as shown in figure 4. For the fluid addition into the main stream through the permeable walls, i.e., \( v_w < 0 \), on the other hand, the asymptote \( v_r(r \to \infty, 0) \) takes a positive value, and then there certainly exists the point of \( v_r(0, 0) = 0 \), where we put \( r = r_1 \) at \( v_r = 0 \) and \( v_w = 0 \). This point \( r_0 \) is obtained from (83) to be

\[
r_0 = -\frac{\nu Re}{2v_w} \left( 1 - \frac{1}{56} \alpha Re \right),
\]

where \( \alpha Re < 105/4 \) and \( \alpha Re < 105/4 \). This point \( r_0 \) is obtained from (83) to be

\[
r_0 = -\frac{\nu Re}{2v_w} \left( 1 - \frac{1}{56} \alpha Re \right),
\]

where \( \alpha Re < 105/4 \) and \( \alpha Re < 105/4 \). This point \( r_0 \) is obtained from (83) to be

\[
r_0 = -\frac{\nu Re}{2v_w} \left( 1 - \frac{1}{56} \alpha Re \right),
\]

where \( \alpha Re < 105/4 \) and \( \alpha Re < 105/4 \). This point \( r_0 \) is obtained from (83) to be

Expressing the finite region of the channel as \( r_2 \leq r \leq r_1 \) as shown in figure 2, we can classify the flow into the following three categories:

(i) for \( r_2 < r_0 \), the main flow is in the direction away from the origin, i.e., \( v_r(r, 0) > 0 \) everywhere,

(ii) for \( r_1 < r_0 \), the flow is directed toward the origin, i.e., \( v_r(r, 0) < 0 \) everywhere,

(iii) for \( r_2 \leq r_0 \leq r_1 \), the flow is directed toward the origin, i.e., \( v_r(r, 0) < 0 \) in \( r_2 \leq r < r_0 \) and the flow is in the direction away from the origin, i.e., \( v_r(r, 0) > 0 \) in \( r_0 < r < r_1 \).

Figure 9 shows the change of the radial velocity along the symmetrical axis of the flows with \( v_w = -0.4, -0.2, 0, 0.2, 0.4 \text{ mm s}^{-1} \) where in turn \( |v_w/U| = 0.85, 0.43, 0, 0.43, 0.85 \) for \( Re = 0.1 \) and \( v_w = -1, -0.5, -0.2, 0, 1 \text{ mm s}^{-1} \) where in turn \( |v_w/U| = 0.21, 0.11, 0.043, 0, 0.21 \) for \( Re = 1 \) in the configuration of figure 8(a); although the convergence of the theory is not sufficiently correct for larger values of \( |v_w/U| \) as
discussed in appendix A, numerical results are shown for reference. The abscissa denotes the radial distance \( r \), while the ordinate does the distance from the symmetrical axis. For \( v_w > 0 \), the velocity along the symmetrical axis is negative everywhere in the channel, that is, the flow is directed toward the origin as mentioned above. Only in the condition of \( Re = 0.1 \) and \( v_w = -0.2 \text{ mm s}^{-1} \), the velocity intersects the abscissa at a point. We can get its intersection point to be \( r_0 = 0.25 \text{ mm} \) from (85), it is also recognized in the left-hand side of figure 9. The flow is in the direction away from the origin everywhere for \( Re = 0.1 \) and \( v_w = -0.4 \text{ mm s}^{-1} \).

7. Conclusions

The steady two-dimensional microscopic and laminar flows between two intersecting permeable walls were analyzed in a theoretical way. The stream function was found so as to suitably combine the two unknown functions describing the main stream flow and the permeable flow. The perturbation method was adopted to seek solutions of these governing equations, and the radial and angular velocity components of the main flow up to the first order corrections to them for the convergent angle were obtained. The friction loss of the main flow with the zeroth-order solutions derived, and the ratio of Reynolds number of the permeable flow to that of the main flow was introduced as the parameter which represents the effect of the permeable flow on the main flow. It was demonstrated that the friction loss of the main flow is larger that of the so-called Jeffery–Hamel flow for the convergent flow in the finite length of channel. Furthermore, the present theory predicted that the main flow with fluid addition through permeable walls was inversely directed away from the origin due to the balance of the main flow and the permeable one, while the one with fluid suction through permeable walls was directed toward the origin in all conditions. The extension of the present theory to an axisymmetric flow in a permeable tube, such as the flow of the tube radius changing linearly along the axis, is easy, and is now in progress.

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Appendix A. Scaling of all terms in (23) and (24)

Let us show that the third term of the right-hand side of (23) is much smaller than the other terms. In the following, we suppose for simplicity the convergent flow \((Q_0 < 0)\) in the finite length of the two-dimensional channel between the radius \( r_1 \) at the position 1 and the radius \( r_2 \) at the position 2, as shown in figure 2. Then the reference velocity \( U(>0) \) can be defined by \( Q_0, \rho \) and \( r \) as

\[
U = -\frac{Q_0}{\rho r},
\]

(A.1)

where \( r \) is the reference length defined by the harmonic mean of the radius \( r_1 \) and the radius \( r_2 \), i.e., \( r = 2r_1r_2/(r_1 + r_2) \), which was used in equation (77). Dimensionless length and
pressure with the superscript "∗" are introduced to be

\[ r^* = \frac{r}{\bar{r}}, \quad p^* = \frac{p}{\rho U^2}. \]  

(A.2)

With use of (A.1) and (A.2), the equations (23) and (24) can be written in dimensionless forms as

\[
\frac{\partial p^*}{\partial r^*} = \frac{1}{r^*} \left[ \frac{1}{2} \left( g'' \right)^2 + \frac{1}{\text{Re}_\alpha} \left( f' g' \right) + \frac{1}{r^2 U} \left( \frac{1}{2} \alpha^2 \left( f'' + f' \right) \right) \right],
\]

\[
\frac{\partial p^*}{\partial \lambda} = \frac{1}{r^* \text{Re}_\alpha} (g'' + \frac{1}{r^* U} \left[ \frac{1}{2} f' g' + \frac{1}{\alpha} \left( f'' + \alpha f' \right) \right]),
\]

(A.3)

(A.4)

where \( \text{Re} \) is defined to be (Schlichting 1968)

\[ \text{Re} = \frac{U \bar{r}}{\nu}. \]  

(A.5)

Reynolds number \( \text{Re} \) redefined by (A.5) is the same as its definition given by equation (29), but it makes easy to estimate the order of all terms of the right-hand side of (A.3) and (A.4). In case where the magnitude of the main flow velocity is much larger than that of the permeable flow, the condition \( (\nu_w / U)^2 \ll 1 \) is satisfied. Then, the third term of the order of \( (\nu_w / U)^2 \) in (A.3), i.e., the third term of the equation (23) in section 3.1, can be discarded.

Appendix B. Comparison of magnitudes of the leading and correction terms of \( g_1(\lambda) \) and \( f_1(\lambda) \)

Here, let us seek out the second-order correction terms of \( g_2(\lambda) \) and \( f_2(\lambda) \), and evaluate their magnitudes in order to assure the convergence of series expansions of the functions. From the differential equation (36) and its boundary conditions (40), (41), (44) and (46), \( g_2(\lambda) \) can be obtained as

\[
g_2(\lambda) = \frac{1}{10} \lambda (1 - \lambda^2)^2 + \frac{1}{1724 800} \text{Re}^2 \lambda (1 - \lambda^2)^2 (2875 - 2472 \lambda^2 + 959 \lambda^4 - 98 \lambda^6).
\]

(B.1)

Furthermore, using \( g_0, f_0, g_1, f_1 \) and equation (39), which all have already been determined, enables us to get

\[
f_2^{(4)}(\lambda) = 6 \lambda + \frac{9}{1120} \text{Re}^2 \lambda (205 - 830 \lambda^2 + 721 \lambda^4 - 144 \lambda^6).
\]

(B.2)

By direct integration of (B.2) under the \( f_2 \)'s boundary conditions (45), (47)–(49), \( f_2(\lambda) \) can be determined to be

\[
f_2(\lambda) = \frac{1}{20} \lambda (1 - \lambda^2)^2 + \frac{1}{20 697 600} \text{Re}^2 \lambda (1 - \lambda^2)^2 (62 319 - 94 102 \lambda^2 + 33 607 \lambda^4 - 3024 \lambda^6).
\]

(B.3)
Figure B1 shows the magnitudes of the zeroth-order terms, and the first- and second-order correction terms of $g(\lambda)$ and $f(\lambda)$ for $Re = 1$ and $\alpha = 32^\circ$.

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