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非線形シュレディンガー方程式における ノイズによる正則化

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- ノイズと非線形偏微分方程式

(従来の見方）

ノイズ $\Longrightarrow$ 滑らかさが少ない

$\Longrightarrow$ 解に良くない影響

(新しい見方)

ノイズは良い影響を及ぼすこともある $\Longrightarrow$

ノイズを加えることにより，解の一意性，解の大域存在が回復(?) *Regularization by Noise*

例．解の一意性の回復

$$du = \sqrt{|u|}dt + d\beta, \qquad u(0) = 0.$$

- 1D quintic NLS with white noise dispersion

$$idu + \partial_x^2 u \circ d\beta(t) = \lambda |u|^4 u \; dt, \qquad (1)$$

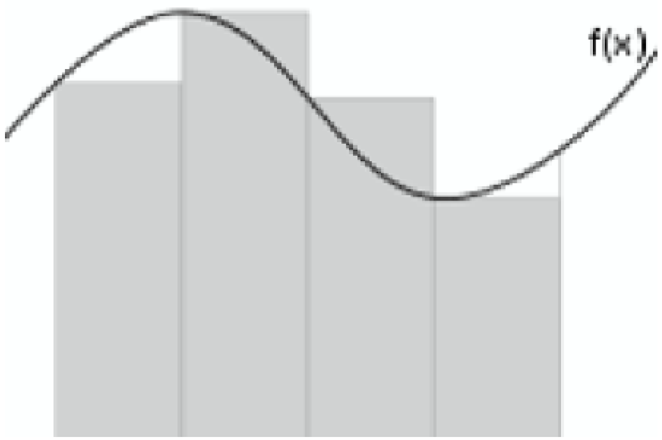
$$t \; > 0, \quad x \in \mathbf{R}$$

$$u(0, x) = u_0(x). \qquad (2)$$

u ; slowly varying envelope of electiric field,
$\beta(t)$; real Brownian motion starting at 0 with mean 0,
$\lambda = 1$ (defocusing) or $\lambda = -1$ (focusing)
$\partial_x^2 u \circ d\beta(t)$; Stratonovich product

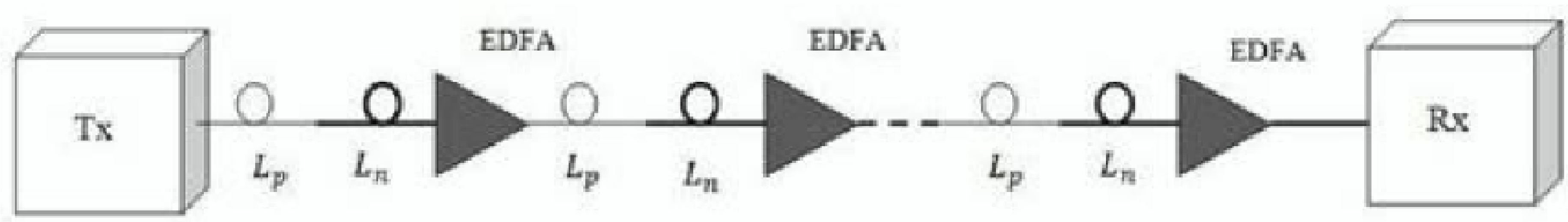
Remark 1 Stratonovich integral ensures the L^2 norm conservation, though the L^2 norm is not conserved by the Itô integral. Both of them are regarded as a stochastic version of Riemann-Stieltjes integral.

$$\sum_{j=1}^{N} f(t_j^*, \omega)\big(\beta(t_j, \omega) - \beta(t_{j-1}, \omega)\big).$$



- In the context of nonlinear fiber optics, variable t corresponds to the distance along a fiber and variable x corresponds to the time. But we keep the conventional notation.
- Equation (1) may be thought of as the diffusion limit of NLS with random dispersion, which describes the propagation of a signal in an optical fiber with managed dispersion.

The carrier fiber has positive dispersion value. For dispersion control, one arranges negative dispersion fibers between positive fibers so that the overall distortion is very small.



分散マネジメントファイバ構成例 (NEC)

http://www.nec.co.jp/techrep/ja/journal/g09/n04/090403-75.html

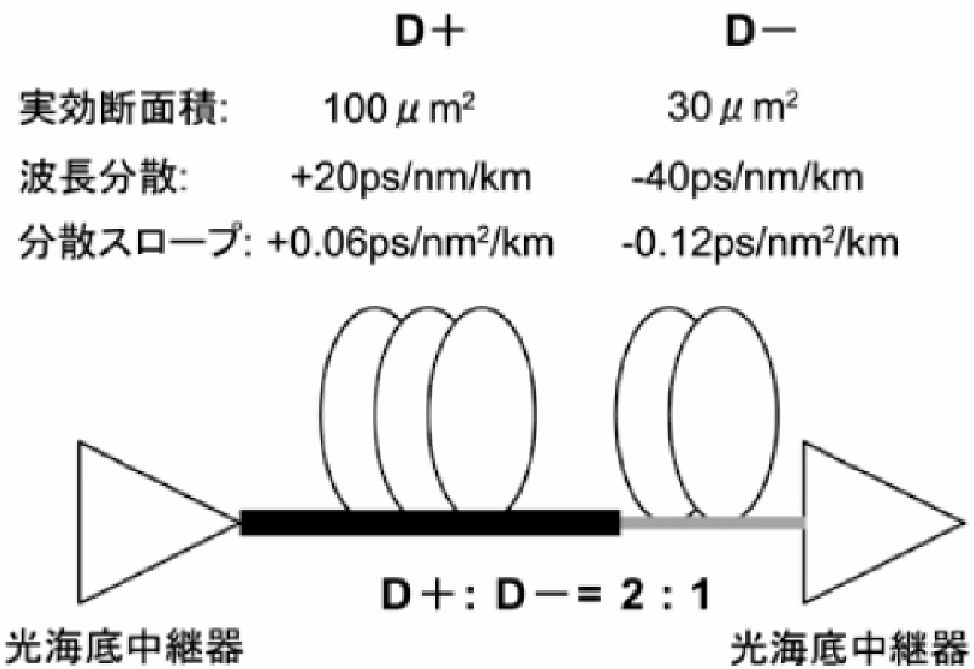


図.1 分散マネジメントファイバ構成例

Microscopic Model Equation

$$i\partial_t u + \frac{1}{\varepsilon} m\big(t/\varepsilon^2\big)\partial_x^2 u = \lambda |u|^4 u. \qquad (3)$$

$m(t)$; stationary (continuous or smooth) random process with zero mean,
$\varepsilon > 0$; size of correlation length for $\frac{1}{\varepsilon} m(\frac{t}{\varepsilon^2})$,
Limit as $\varepsilon \to 0$ is called "diffusion limit" or "diffusion approximation"

For more details of physical model, see the following two books:

G.P. Agrawal, "*Nonlinear Fiber Optics*", 3rd Edition, Academic Press, 2001.

G.P. Agrawal, "*Applications of Nonlinear Fiber Optics*", Academic Press, 2001.

- Deterministic Case ($\partial_x^2 \circ d\beta$ is replaced by ∂_x^2)
  $\lambda = 1$ (defocusing) Dodson, 2010
  $\forall u_0 \in L^2 \implies \exists$ unique global solution u of (1)-(2) in $C([0,\infty); L^2)$.

  $\lambda = -1$ (focusing)
  $\forall u_0 \in L^2 \implies \exists T > 0$ and $\exists$ unique local solution u of (1)-(2) in $C([0,T); L^2)$.
  $\exists u_0 \in H^1$; solution u blows up in finite time.

  T. Cazenave, "*Semilinear Schrödinger Equations*", Amer. Math. Soc., 2003.

Remark 2

$$i\partial_t u + \Delta u = \lambda |u|^{p-1} u, \quad x \in \mathbf{R}^d \qquad (4)$$

Scaling $u_\eta(t,x) = \eta^{2/(p-1)} u(\eta^2 t, \eta x)$, $\eta > 0$ leaves equation (4) invariant, and when $p = 1 + 4/d$, L^2 norm of u_η is also invariant.

$p = 1 + 4/d$; L^2-critical
E.g., $d = 1$ and $p = 5 \implies$ 1 D quintic NLS,

$1 < p < 1 + 4/d$; L^2-subcritical

- (deterministic Strichartz' estimate)
  $d = 1$, $U(t) = e^{it\partial_x^2}$, $T > 0$,

$$\left\|U(\cdot)u_0\right\|_{L^r([0,T];L^q)} \leq CT^\alpha \|u_0\|_{L^2}, \quad (5)$$

$$\alpha := \frac{2}{r} - \left(\frac{1}{2} - \frac{1}{q}\right) \geq 0, \quad 2 \leq q, r \leq \infty. \quad (6)$$

Remark 3 (i) Strichartz' estimate (5) with $\alpha = 0$ is needed for the proof of unique local existence theorem of solution for the L^2 critical case. In the L^2-subcritical case, (5) with $\alpha > 0$ is sufficient.

(ii) In the L^2 subcritical case, existence time T can be estimated by the size of L^2 norm of initial data only. But this is not the case with 1 D quintic NLS because of its L^2 criticality.

- Stochastic Case

Theorem 1 (Debussche-Y.T, 2011)
$\forall u_0 \in L^2$, $\exists$ unique global solution u of (1)-(2) satisfying $u \in C([0,\infty); L^2)$ $a.s.$ and $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, $t > 0$ $a.s.$

Remark 4 In sharp contrast to the deterministic case, 1 D quintic NLS (1) with white noise dispersion has global solution a.s. for each $u_0 \in L^2$, no matter whether λ is negative or positive: *Regularization by Noise*

- Known Results
  R. Marty, 2006, Equation (1) with nonlinearity $f(|u|^2)u$, $f \in C_b^1([0,\infty);\mathbf{R})$, $\forall u_0 \in L^2 \implies \exists$ unique global solution of (1)-(2) a.s.

A. de Bouard and A. Debussche, 2010, nonlinearity $\lambda|u|^{p-1}u$, $1 < p < 1 + 4/d$, $\forall u_0 \in L^2 \implies \exists$ unique global solution u of (1)-(2) satisfying $u \in C([0,\infty); L^2)$ a.s. and $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, $t > 0$ a.s.

Remark 5 The L^2 critical case is excluded in the paper by de Bouard and Debussche. Their proof is based on a stochastic version of Strichartz' estimate. But their Strichartz estimate does not cover the case

$$\alpha := \frac{2}{r} - \left(\frac{1}{2} - \frac{1}{q}\right) = 0$$

- Difficulties to prove stochastic Strichartz

  Let $s \geq 0$ be fixed.

$$i du + \Delta u \circ d\beta(t) = 0, \quad t > s, \; x \in \mathbf{R}^d,$$
$$u(s) = u_s.$$

$$U(t,s)u_s = \mathcal{F}^{-1}\left[e^{i|\xi|^2(\beta(t)-\beta(s))}\hat{u}_s(\xi)\right],$$
$$t \geq s \geq 0.$$

1. $Z_{s,\varepsilon} = \{s + \varepsilon \geq t \geq s |\ \beta(t) - \beta(s) = 0\}$,
$s > 0$, $\varepsilon > 0$,
$Z_{s,\varepsilon}$ has the cardinality of continuum a.s.

2. Duality argument does not work as well as in the deterministic case.

Lemma 1 (Debussche-Y.T)

$$\mathbb{E}\left[\|\int_0^t U(t,s)f(s)\ ds\|^4_{L^5([0,T];L^{10})}\right] \qquad (7)$$
$$\leq CT^{2/5}\mathbb{E}\left[\|f\|^4_{L^1([0,T];L^2)}\right], \quad T > 0.$$

Remark 6 Lemma 1 almost corresponds to the case $\alpha := \frac{2}{r} - \left(\frac{1}{2} - \frac{1}{q}\right) = 0$ with $q = 10$ and $r = 5$, compared with deterministic Strichartz (5). Moreover, the power of T is positive in Lemma 1, while $\alpha = 0$ in the determinisitic case. In this respect, the stochastic Strichartz estimate is better than the deterministic one.

bilinear (or sesquilinear) Strichartz estimate by Ozawa and Y.T (1998) $\Longrightarrow$ Lemma 1

- Bilinear Strichartz Estimate
  Sesquilinear mapping:

$$(u_0, v_0) \longmapsto \left(e^{it\partial_x^2} u_0\right) \overline{\left(e^{it\partial_x^2} v_0\right)}$$

Lemma 2 (Ozawa-Y.T, 1998)

$$\left\| \left(-\partial_x^2\right)^{1/4} \left[\left(e^{it\partial_x^2} u_0\right) \overline{\left(e^{it\partial_x^2} v_0\right)} \right] \right\|_{L^2(\mathbf{R}\times\mathbf{R})} \leq C \|u_0\|_{L^2} \|v_0\|_{L^2}. \tag{8}$$

Remark 7 Estimate (8) fails if the term

$$\left(e^{it\partial_x^2}u_0\right)\overline{\left(e^{it\partial_x^2}v_0\right)}$$

is replaced by

$$\left(e^{it\partial_x^2}u_0\right)\left(e^{it\partial_x^2}v_0\right)$$

on the left hand side of (8). Estimate (8) implies the smoothing effect of Schrödinger equation.

- Stochastic Version of Bilinear Strichartz Estimate

Lemma 3

$$\mathbb{E}\left[\int_0^T \left\| D^{1/2}\left(\left|\int_0^t U(t,s)f(s)ds\right|^2\right)\right\|_{L^2(\mathbf{R})}^2 dt\right] \leq 4\sqrt{2\pi}\, T^{1/2}\mathbb{E}\left[\|f\|_{L^1(0,T;L^2(\mathbf{R}))}^4\right].$$

Remark 8 Factor $T^{1/2}$ appears on the right hand side of the above inequality, which is the difference between the deterministic and the stochastic bilinear Strichartz estimates. The integration with respect to variable ω of probablity space $(\Omega, \mathcal{B}, P)$ yields a kind of stochastic smoothing effect. For other equations, see [Flandori, LNM **2015** (2011)].

Lemma 3 + Sobolev $\implies$ Lemma 1
Lemma 1 + Contraction $\implies$ Theorem 1

- Diffusion Limit
  (Macroscopic Model)

$$idu + \partial_x^2 u \circ d\beta(t) = \lambda |u|^4 u \; dt. \quad (1)$$

(Microscopic Model)

$$i\partial_t u + \frac{1}{\varepsilon} m\big(t/\varepsilon^2\big) \partial_x^2 u = \lambda |u|^4 u. \quad (3)$$

Problem: As $\varepsilon \to +0$, does the solution of (3) converge to the solution of (1)?

Theorem 2 (Debussche and Y.T, 2011)
$m(t)$; continuous random process, "*ergodic*".
$\forall \varepsilon > 0$, $\forall u_0 \in H^1$, $\exists\ \tau_\varepsilon(u_0) > 0$, $\exists$ unique solution u_ε of (3) in $C([0, \tau_\varepsilon(u_0)); H^1)$ a. s.
Moreover, for any $T > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{P}(\tau_\varepsilon(u_0) \leq T) = 0,$$

$$u_\varepsilon \mathbf{1}_{\{\tau_\varepsilon > T\}} \longrightarrow u \quad (\varepsilon \to 0)$$

$$\text{in distribution of } C([0, T]; H^1),$$

where u is the solution of (1).

Remark 9 We note that Theorem 2 holds for nonlinearity $\lambda|u|^{p-1}u$, $1 < p \leq 5$. Even in the subcritical case $p < 5$, Theorem 2 is an improvement over previous results concerning the topology of convergence:

Marty, 2003,
$f(|u|^2)u, \ f \in C_b^1([0,\infty);\mathbf{R}), \ u_0 \in H^2$
$\Longrightarrow \ u_\varepsilon \ \to \ u \ (\varepsilon \to 0)$
in distribution of $C([0,T];H^2)$.

de Bouard and Debussche, 2010,

$p < 5, \quad u_0 \in H^1 \implies u_\varepsilon \to u \quad (\varepsilon \to 0)$

in distribution of $C([0,T]; H^s)$, $s < 1$.

Remark 10 The ergodic assumption on m ensures that

$$\int_0^t \varepsilon^{-1} m(\varepsilon^{-2} s)\, ds \longrightarrow \beta(t) \quad (\varepsilon \to 0)$$

$$\text{in distribution of } C([0,T]; \mathbf{R}), \quad \forall T > 0.$$

$m(t)$, $m(s)$, $t > s$ *need not* be independent.

- M. Rosenblatt, A central limit theorem and a strong mixing condition, Proc. Nat. Acad. Sci. U.S.A., **42** (1956), 43–47.
- P. Billingsley, “Convergence of Probability Measures”, 2nd Edition, 1999.
(see, Chapter 5)