解の表示式と大域的分岐構造について

Limiting equation of a cross-diffusion equation　の定常解

Cahn-Hilliard equation の定常解

cross-diffusion equation
limiting equation as $r \to \infty$

$$
(S)\quad \begin{cases} \int_0^1 \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right)dx = 0, \\ d_2 v_{xx} + v\left(a_2 - b_2\frac{\tau}{v} - c_2 v\right) = 0 \ \ in \ \ (0,1), \\ v_x(0) = v_x(1) = 0, \\ v > 0 \ \ in \ \ (0,1) \end{cases}
$$

$\tau > 0$: unknown constant

$v(x)$: unknown function

We may consider the case

$b_1=1$, $a_2=b_2=c_2=1$.

$\tau > 0$: unknown constant , $v(x)$: unknown

$$
(S)\quad \begin{cases} \displaystyle\int_0^1 \frac{\tau}{v}\left(a_1 - \frac{\tau}{v} - c_1 v\right)dx = 0, \\ d_2 v_{xx} + v\left(1 - \frac{\tau}{v} - v\right) = 0 \ \ in \ \ (0,1), \\ v_x(0) = v_x(1) = 0, \\ v > 0 \ \ in \ \ (0,1), \quad v_x > 0 \ \ in \ \ (0,1) \end{cases}
$$

$\tau > 0$: unknown constant , $v(x)$: unknown

$$
\begin{cases}
d_2 v_{xx} + v - \tau - v^2 = 0, \ \ in(0,1), \\
v_x(0) = v_x(1) = 0, \\
v(x) > 0, in(0,1)
\end{cases}
$$

$$
a_1 = \left(-\int_0^1 \frac{1}{v^2} dx \cdot \tau + c_1 \right) \Big/ \int_0^1 \frac{1}{v} dx
$$

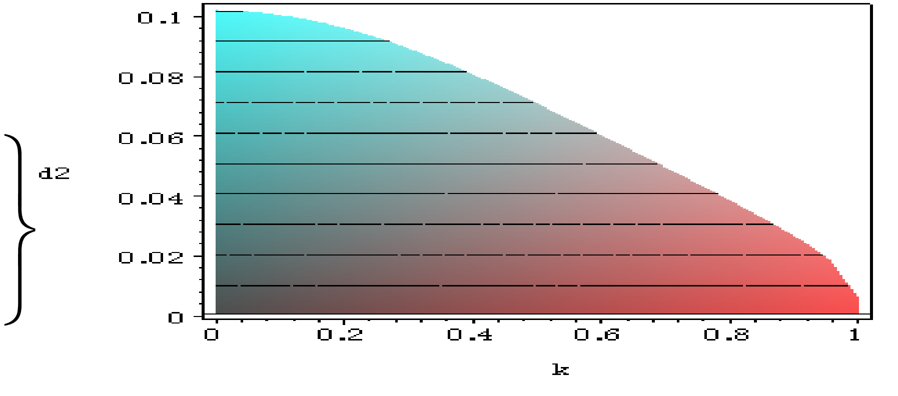
$$\begin{cases} d_2 v_{xx} + v - \tau - v^2 = 0 \ \ in \ \ (0,1), \ \ v_x(0) = v_x(1) = 0, \\ v(x) > 0 \ in \ \ (0,1), \ \ \ v_x(x) > 0 \ in \ \ (0,1) \end{cases}$$

$$v(x;d_2,k) = \frac{1}{2} - 2d_2(k^2+1)K(k)^2 + 6d_2k^2K(k)^2 sn^2\big(K(k)x,k\big),$$

$$\tau(d_2,k) = -\frac{1}{4} + 4(k^4 - k^2 + 1)d_2^2 K(k),$$

where $(k,d_2) \in$

$$\left\{ (k,d_2) : 0 < d_2 < \frac{1}{4(k^2+1)K(k)^2} \right\}$$



$$\begin{cases} d_2 v_{xx} + v - \tau - v^2 = 0, \quad in \ (0,1), \\ v_x(0) = v_x(1) = 0, \\ v(x) > 0 \ in \ (0,1) \end{cases}$$

Exact solution v(x)

$$v(0) := \alpha, \quad v(1) := \beta \quad (\alpha < \beta)$$

Multiply the equation by v_x, and integrate over [0,x] , and use $v_x(1) = 0$, we obtain

$$\frac{dv}{dx} = \sqrt{\frac{1}{d_2}\left(\frac{2}{3}v^3 - v^2 - 2\tau v - \frac{2}{3}\alpha^3 + \alpha^2 + 2\tau\alpha\right)}$$

Substitute x = 1

$$0 = \sqrt{\frac{1}{d_2}\left(\frac{2}{3}\beta^3 - \beta^2 - 2\tau\beta - \frac{2}{3}\alpha^3 + \alpha^2 + 2\tau\alpha\right)}$$

$$\frac{2}{3d_2}\left(v^3-\frac{3}{2}v^2-3\tau v-\alpha^3+\frac{3}{2}\alpha^2+3\tau\alpha\right)=0$$

α, β : *real*

the third root γ *becomes real*

$$\frac{dv}{dx}=\sqrt{\frac{2}{3d_2}(v-\alpha)(\beta-v)(\gamma-v)}$$

$$(\alpha<\beta<\gamma)$$

$$\alpha+\beta+\gamma=\frac{3}{2}$$

$$\frac{dv}{dx} = \sqrt{\frac{2}{3d_2}(v-\alpha)(\beta-v)(\gamma-v)}$$

integrate

$$x = \sqrt{\frac{3d_2}{2}}\int_\alpha^v \frac{dw}{\sqrt{(w-\alpha)(\beta-w)(\gamma-w)}}$$

$$w := \alpha + (\beta-\alpha)z^2$$

$$\sqrt{\frac{\gamma-\alpha}{6d_2}}\cdot x = \int_0^{\sqrt{\frac{v-\alpha}{\beta-\alpha}}} \frac{dz}{\left(1-z^2\right)\left(1-\frac{\beta-\alpha}{\gamma-\alpha}z^2\right)}$$

$$k := \sqrt{\frac{\beta-\alpha}{\gamma-\alpha}}$$

$$\sqrt{\frac{\gamma-\alpha}{6d_2}}\cdot x = \int_0^{\sqrt{\frac{v-\alpha}{\beta-\alpha}}} \frac{dz}{\sqrt{\left(1-z^2\right)\left(1-k^2z^2\right)}}$$

substitute $x=1, v=\beta$

$$\sqrt{\frac{\gamma-\alpha}{6d_2}} = K(k)$$

$$\begin{cases} \alpha = \frac{1}{2} - 2d_2K(k)^2\left(k^2+1\right) \\ \beta = \frac{1}{2} + 2d_2K(k)^2\left(2k^2-1\right) \\ \gamma = \frac{1}{2} + 2d_2K(k)^2\left(2-k^2\right) \end{cases}$$

$$\alpha+\beta+\gamma = \frac{3}{2}$$

$$K(k)x = \int_0^{\sqrt{\frac{v-\alpha}{\beta-\alpha}}} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

$$\mathrm{sn}^{-1}(z,k) := \int_0^z \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}$$

$$K(k)x = \mathrm{sn}^{-1}\left(\sqrt{\frac{v-\alpha}{\beta-\alpha}}, k\right)$$

$$\mathrm{sn}(K(k)x, k) = \sqrt{\frac{v-\alpha}{\beta-\alpha}}$$

$$v(x;k,d_2) = \alpha + (\beta-\alpha)\cdot \mathrm{sn}^2(K(k)x, k)$$

τ

$$\frac{2}{3d_2}\left(v^3 - \frac{3}{2}v^2 - 3\tau v - \alpha^3 + \frac{3}{2}\alpha^2 + 3\tau\alpha\right) = 0$$

$\alpha,\ \beta,\ \gamma : real\ \ roots$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -3\tau$$

$$\tau = -\frac{1}{3}\left(\alpha\beta + \beta\gamma + \gamma\alpha\right)$$

$$\begin{cases} \alpha = \frac{1}{2} - 2d_2 K(k)^2\left(k^2 + 1\right) \\ \beta = \frac{1}{2} + 2d_2 K(k)^2\left(2k^2 - 1\right) \\ \gamma = \frac{1}{2} + 2d_2 K(k)^2\left(2 - k^2\right) \end{cases}$$

a_1

$$a_1 = \frac{-\tau \int_0^1 \frac{1}{v^2}\,dx + c_1}{\int_0^1 \frac{1}{v}\,dx}$$

$$\int_0^1 \frac{1}{v}\,dx = \int_\alpha^\beta \frac{1}{v}\cdot\left(\frac{dx}{dv}\right)\cdot dv$$

$$\frac{dv}{dx} = \sqrt{\frac{2}{3d_2}(v-\alpha)(\beta-v)(\gamma-v)}$$

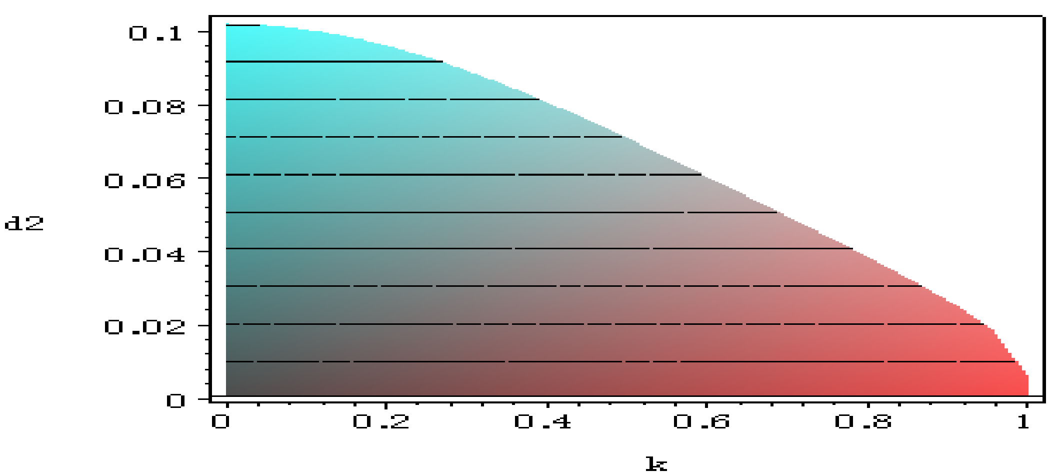
$$\int_0^1 \frac{1}{v}\,dx = \sqrt{\frac{3d_2}{2}}\int_\alpha^\beta \frac{dv}{v\sqrt{(v-\alpha)(\beta-v)(\gamma-v)}}$$

$$\int_0^1 \frac{1}{v^2}\,dx = \int_\alpha^\beta \frac{1}{v^2}\cdot\left(\frac{dx}{dv}\right)\cdot dv$$

$$\frac{dv}{dx} = \sqrt{\frac{2}{3d_2}(v-\alpha)(\beta-v)(\gamma-v)}$$

$$\int_0^1 \frac{1}{v^2}\,dx = \sqrt{\frac{3d_2}{2}}\int_\alpha^\beta \frac{dv}{v^2\sqrt{(v-\alpha)(\beta-v)(\gamma-v)}}$$

$$\tilde{a}_1(k,d_2) := \left\{-3\{4\alpha(\alpha-1)(k^4-k^2+1)-3k^2\}^2\,\Pi\left(\frac{3(1-2\alpha)k^2}{2\alpha(k^2+1)},k\right)\right.$$
$$+2\alpha(k^2+1)(2\alpha k^2-4\alpha+3)$$
$$[4\alpha(4\alpha c_1-\alpha-3c_1)k^4+\{3+4(2c_1+1)\alpha^2$$
$$-4(3c_1+1)\alpha\}k^2-4(2c_1+1)\alpha^2+4\alpha]K(k)$$
$$\left.-6\alpha(2\alpha-1)(k^2+1)\{4\alpha(\alpha-1)(k^4-k^2+1)-3k^2\}E(k)\right\}$$
$$\Big/\ \left\{8\alpha(k^2+1)^2((4\alpha-3)k^2-2\alpha)\right.$$
$$\left.(2(k^2-2)\alpha+3)\,\Pi\left(\frac{3(1-2\alpha)k^2}{2\alpha(k^2+1)},k\right)\right\}$$



where

$$\alpha=\frac{1}{2}-2d_2(k^2+1)K(k)^2$$

Cahn-Hilliard 方程式の定常解の大域的分岐構造

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STATIONARY SOLUTIONS TO THE ONE-DIMENSIONAL CAHN-HILLIARD EQUATION: PROOF BY THE COMPLETE ELLIPTIC INTEGRALS

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(Communicated by Shouchuan Hu)

Abstract. We investigate stationary solutions of the one-dimensional Cahn-Hilliard equation with the diffusion coefficient and the total mass of the density as two given parameters. We solve the equation completely in the whole parameter space by using the Jacobi elliptic functions and complete elliptic integrals. In addition to counting the stationary solutions, which was studied by Grinfeld and Novick-Cohen, we provide an exact expression of the solutions. We also illustrate global bifurcation diagrams together with the asymptotic behavior of the solutions as the diffusion coefficient vanishes.

1. Introduction.

We study the following non-local 2nd order nonlinear differential equation:

$$\begin{cases} \varepsilon^2 \dfrac{d^2u}{dx^2}(x) + f(u(x)) - a = 0, & x \in (0,1), \\ \dfrac{du}{dx}(x) = 0, & x = 0, 1, \\ m = \displaystyle\int_0^1 u(x)\,dx, \quad a = \int_0^1 f(u(x))\,dx, & \end{cases} \tag{1}$$

where ε is a positive parameter and f is the cubic polynomial

$$f(u) := u - u^3.$$

$$
\begin{cases}
\varepsilon^2 \dfrac{d^2u}{dx^2}(x) + f(u(x)) - a = 0, & x \in (0,1), \\
\dfrac{du}{dx}(x) = 0, & x = 0, 1, \\
m = \displaystyle\int_0^1 u(x)\,dx, \quad a = \int_0^1 f(u(x))\,dx,
\end{cases}
\tag{1}
$$

where ε is a positive parameter and f is the cubic polynomial

$$
f(u) := u - u^3.
$$

The above equation is known as the stationary equation of the one-dimensional Cahn-Hilliard equation in the interval $(0,1)$; that is, a solution to (1) gives an equilibrium solution to the Cahn-Hilliard equation. This equation is studied in an extensive literature including [1], [2], [3], [4], [5], [9], and [10]. Among other things, Grinfeld and Novick-Cohen [5] determine the number of solutions to (1) for every (m, ε). They reduced the problem to a system of two equations for two unknown parameters and counted solutions to the system by applying some topological argument, called the transversality argument. From their result we can easily see how the solution bifurcates as (m, ε) varies.

On the other hand it might be nice to obtain an explicit form of each solution to (1) for every (m, ε). Indeed if a solution to (1) has an exact analytical expression,

$$\begin{cases} \varepsilon^2 \dfrac{d^2u}{dx^2}(x) + f(u(x)) - a = 0, & x \in (0,1), \\ \dfrac{du}{dx}(x) = 0, & x = 0, 1, \\ m = \displaystyle\int_0^1 u(x)\,dx, \quad a = \int_0^1 f(u(x))\,dx, \end{cases} \tag{1}$$

where ε is a positive parameter and f is the cubic polynomial

$$f(u) := u - u^3.$$

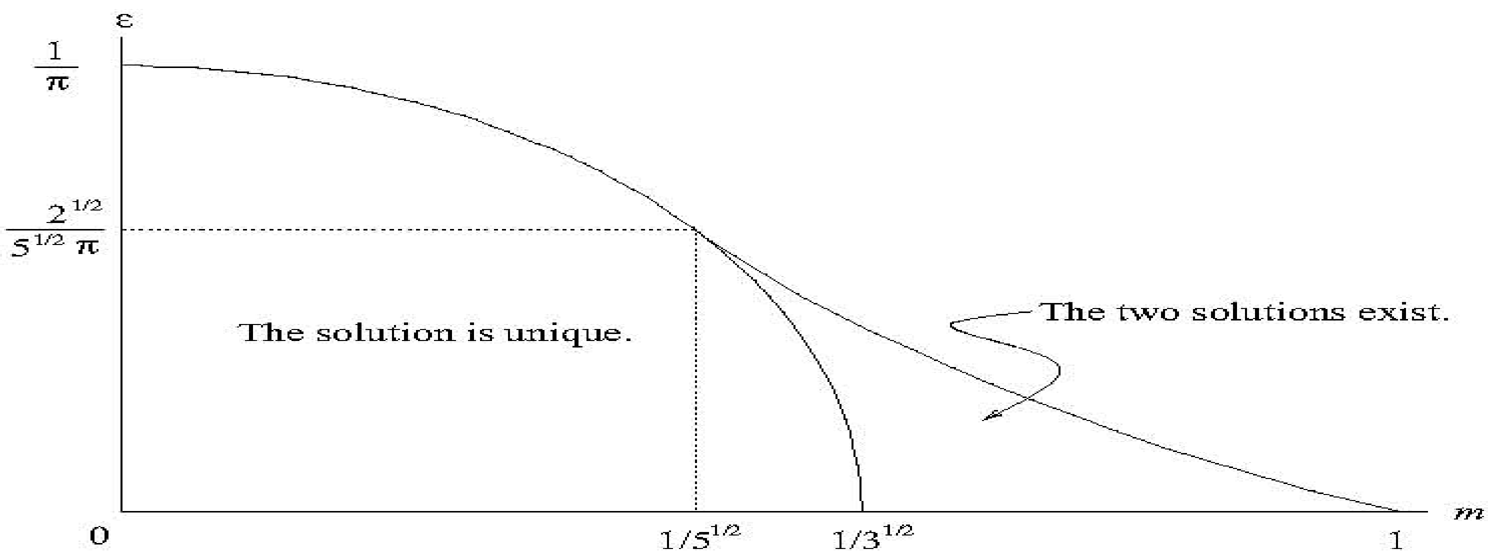
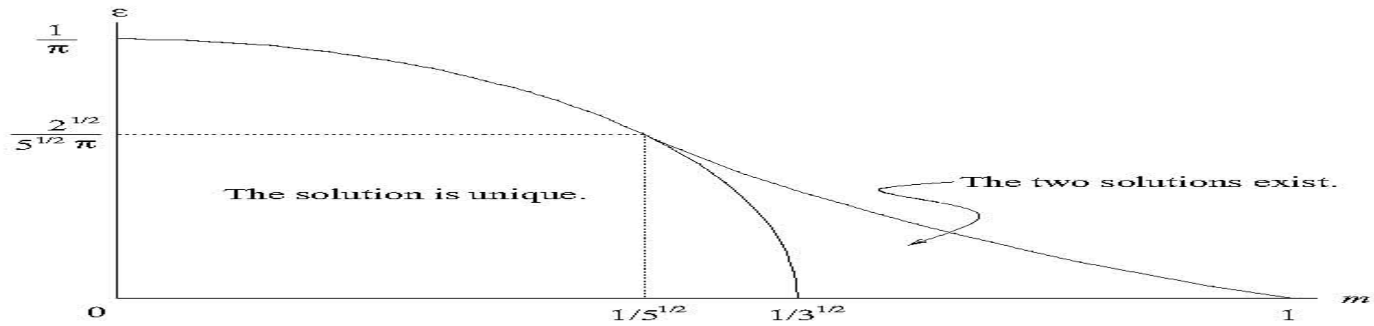


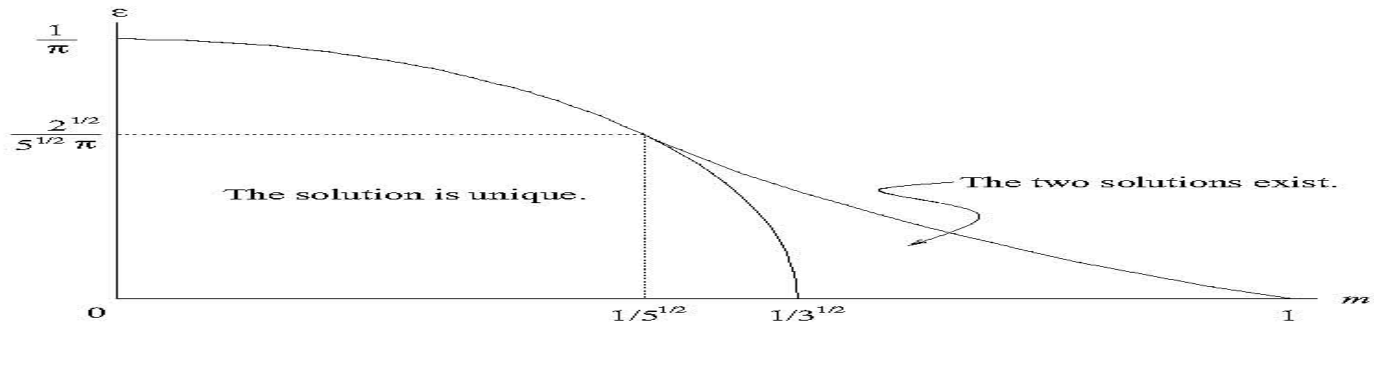
FIGURE 1. The region where (1) has solutions.

Fix ε= 1/4.



ε
$\frac{1}{\pi}$
$\frac{2^{1/2}}{5^{1/2}\pi}$
The solution is unique.
The two solutions exist.
0
$1/5^{1/2}$
$1/3^{1/2}$
1
m

Fix ε= 1/4.


ε
1/π
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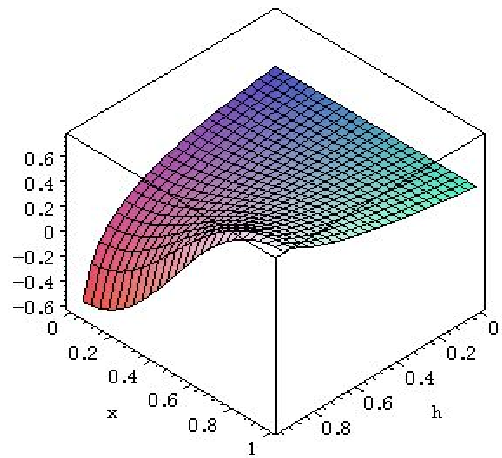
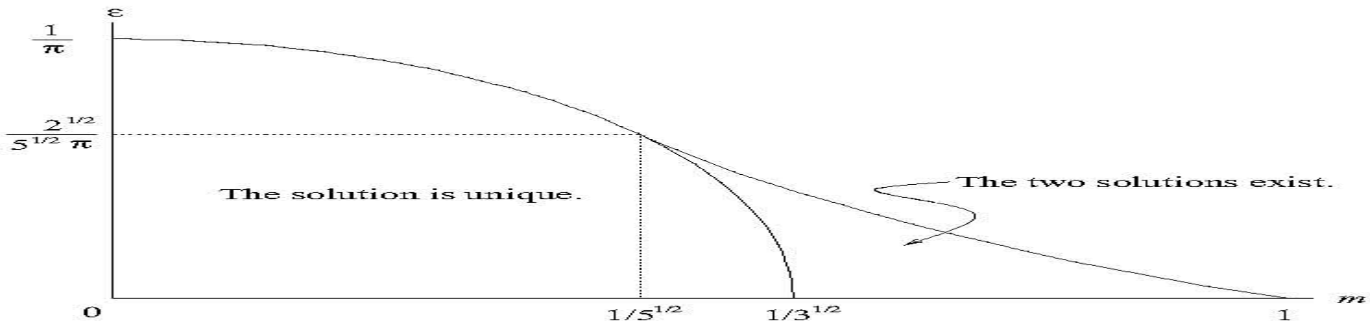
0
1/5^{1/2}
1/3^{1/2}
1
m


0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0
0.2
0.4
0.6
0.8
1
x
0
0.2
0.4
0.6
0.8
h

Fix ε= 1/4.


ε
$\frac{1}{\pi}$
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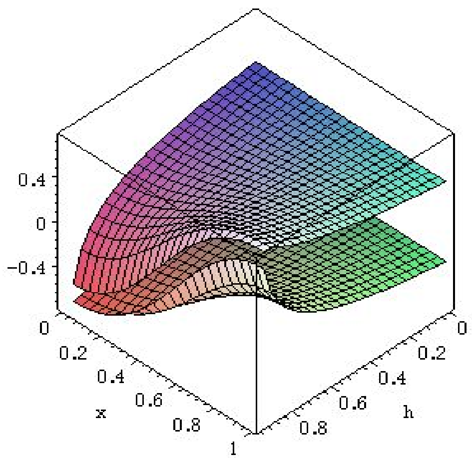
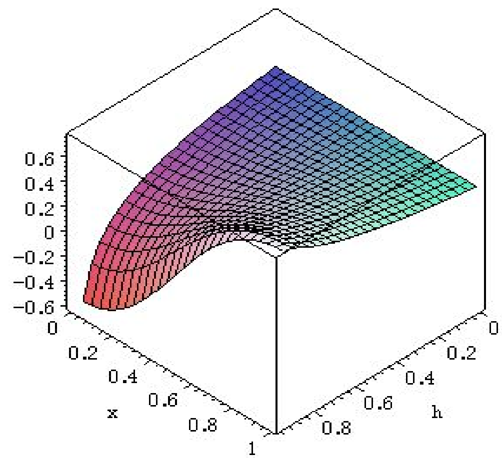
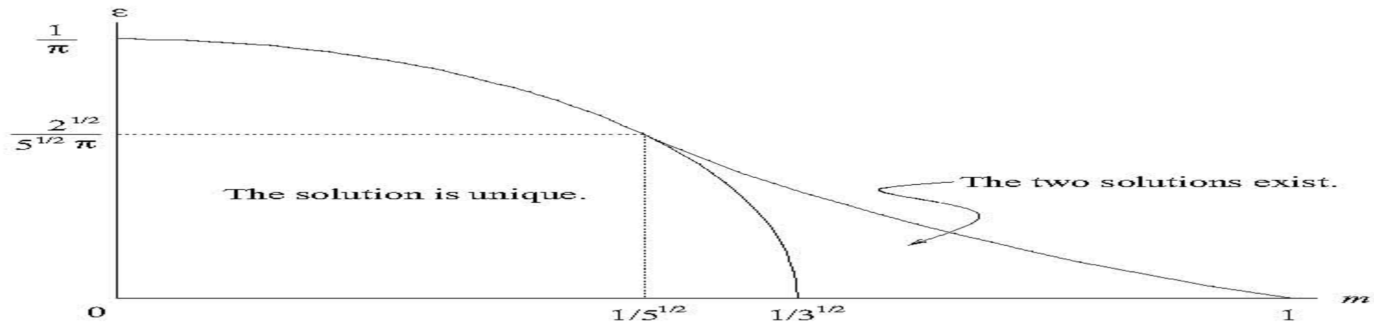
0
$1/5^{1/2}$
$1/3^{1/2}$
1
m


0.4
0
-0.4
0
0.2
0.4
0.6
0.8
1
x
0
0.2
0.4
0.6
0.8
h


0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
0.2
0.4
0.6
0.8
1
x
0
0.2
0.4
0.6
0.8
h

Fix ε= 1/4.



ε
1/π
$\frac{2^{1/2}}{5^{1/2}\pi}$
The solution is unique.
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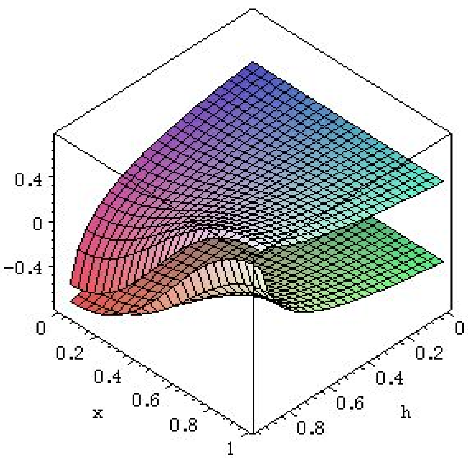
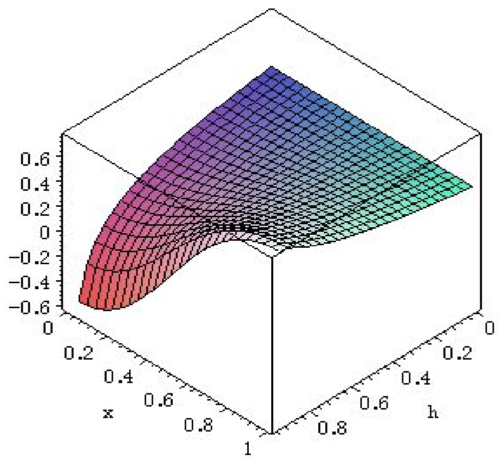
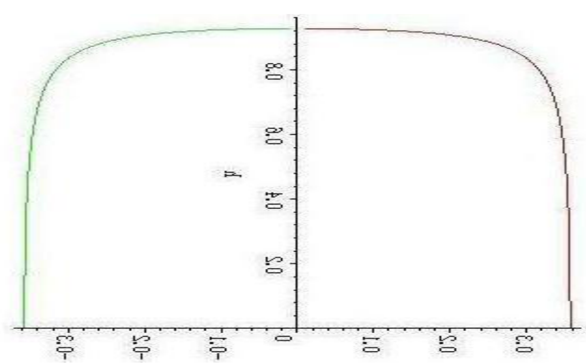
0
$1/5^{1/2}$
$1/3^{1/2}$
1
m



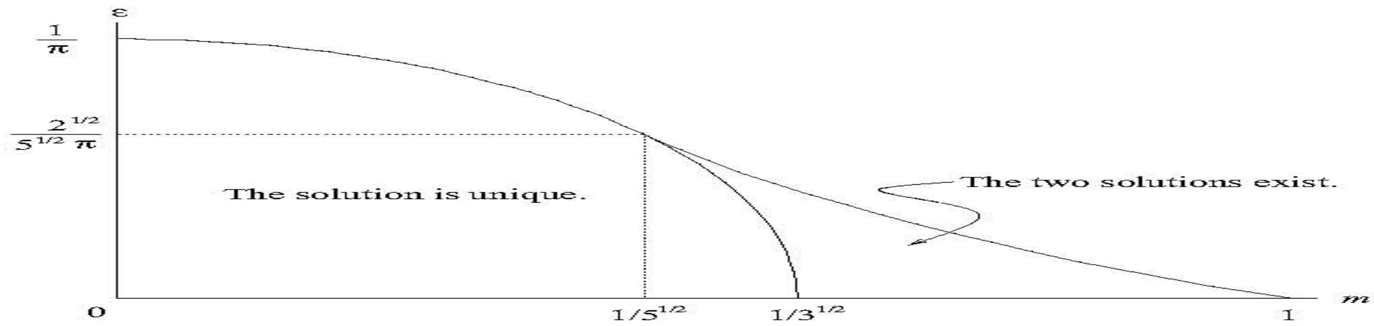
0.4
0
−0.4
0
0.2
0.4
0.6
0.8
1
x
h



0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0
0.2
0.4
0.6
0.8
1
x
h

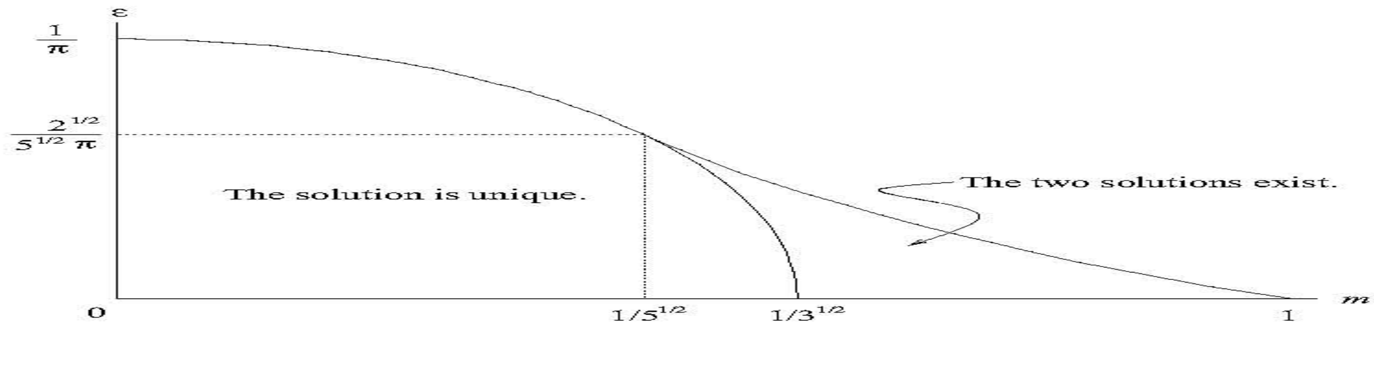


Fix ε= 1/6.



ε
1/π
2^1/2 / 5^1/2 π
The solution is unique.
The two solutions exist.
0
1/5^1/2
1/3^1/2
1
m

Fix ε= 1/6.



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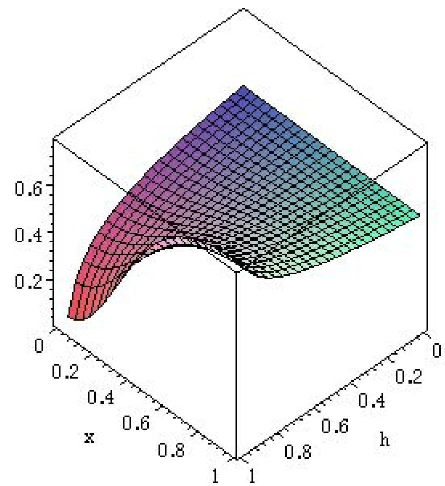
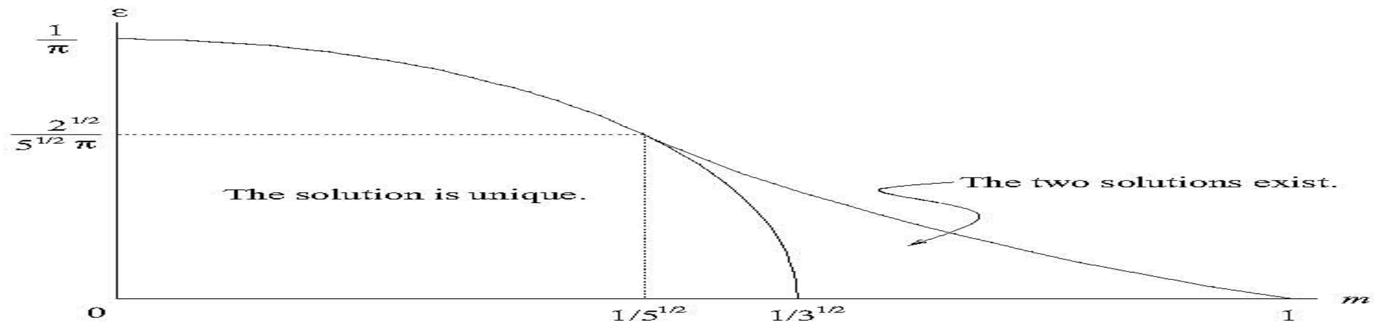
0
1/5^1/2
1/3^1/2
1
m



0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
x
h

Fix ε= 1/6.



ε
1/π
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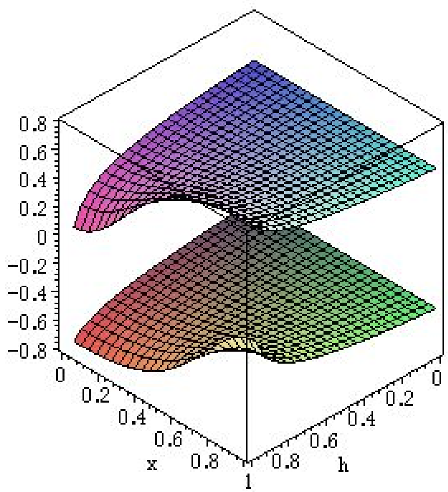
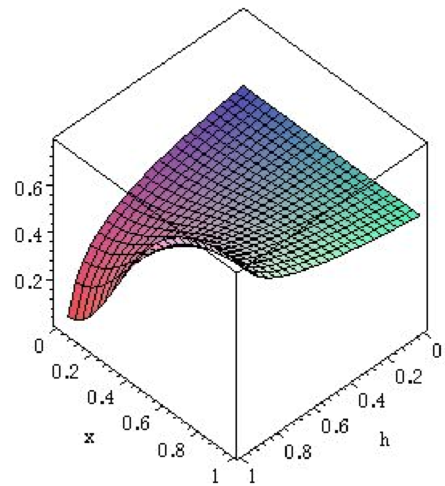
0
1/5^{1/2}
1/3^{1/2}
1
m

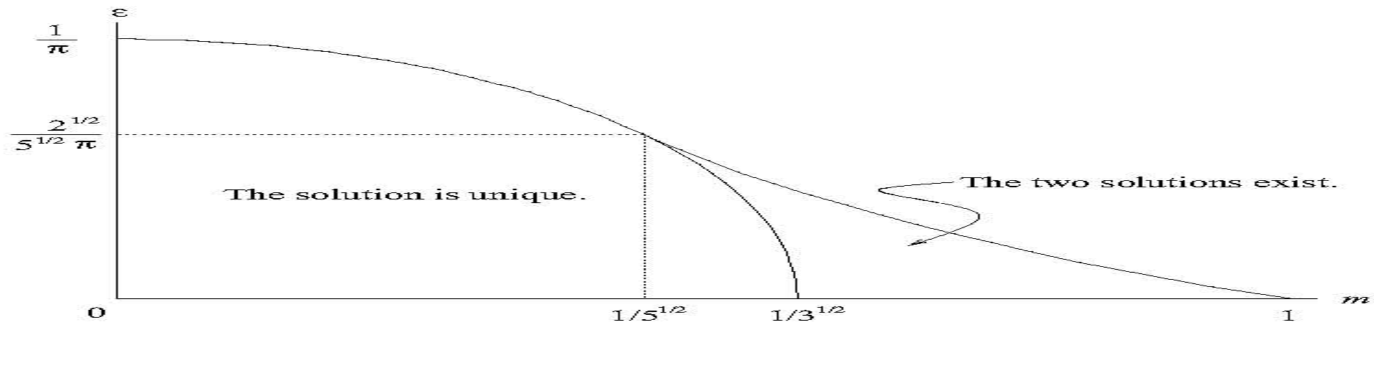


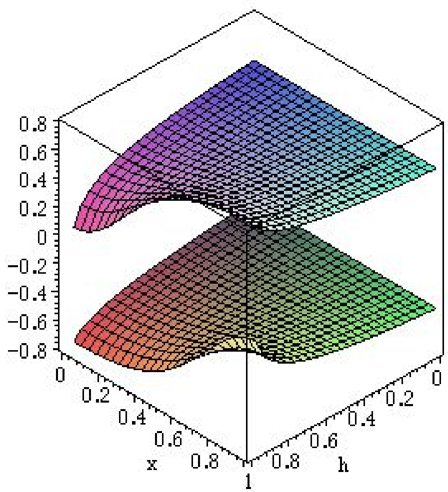
x
h



x
h

Fix ε= 1/6.


ε
1/π
2^1/2 / 5^1/2 π
The solution is unique.
The two solutions exist.
0
1/5^1/2
1/3^1/2
1
m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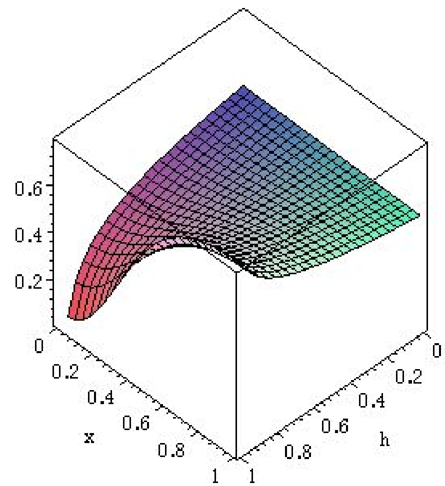
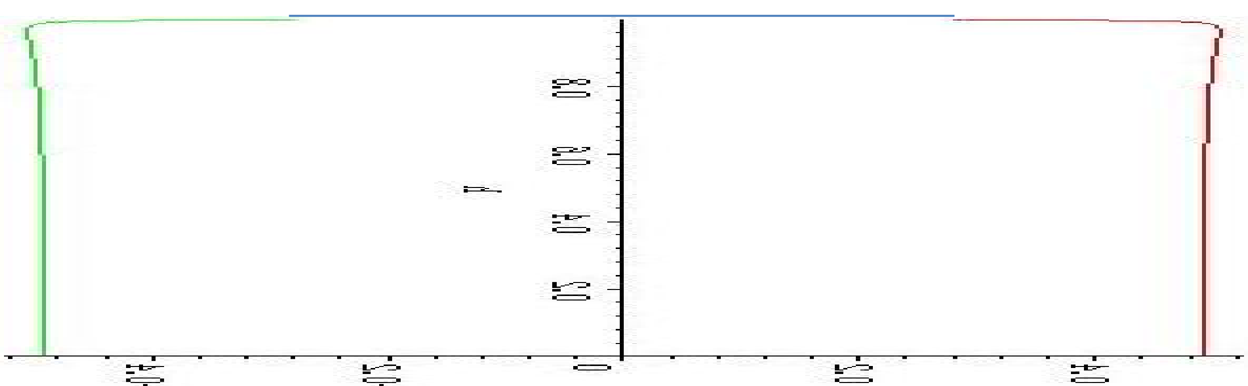
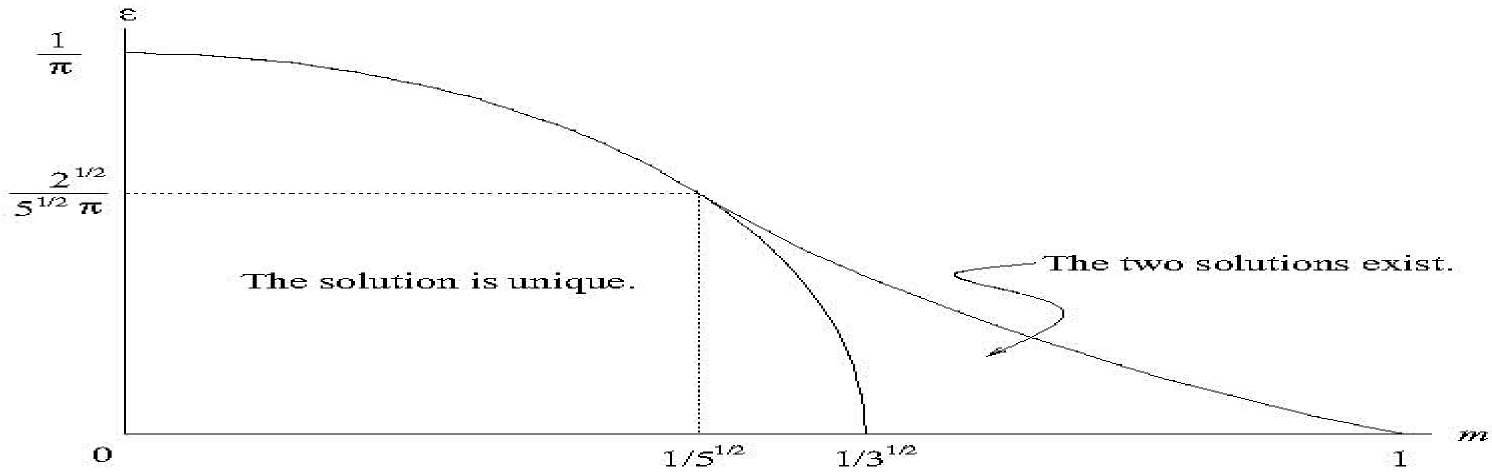
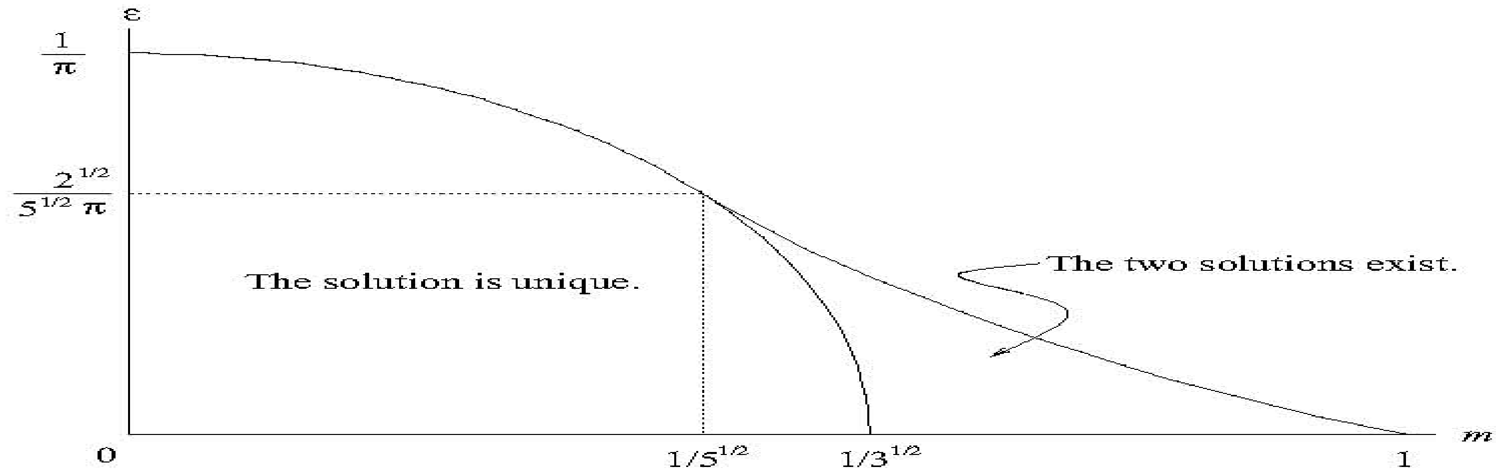
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
x
h


0.6
0.4
0.2
x
h




ε
1/π
2^{1/2}/(5^{1/2} π)
The solution is unique.
The two solutions exist.
0
1/5^{1/2}
1/3^{1/2}
1
m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2^{1/2} / 5^{1/2} π
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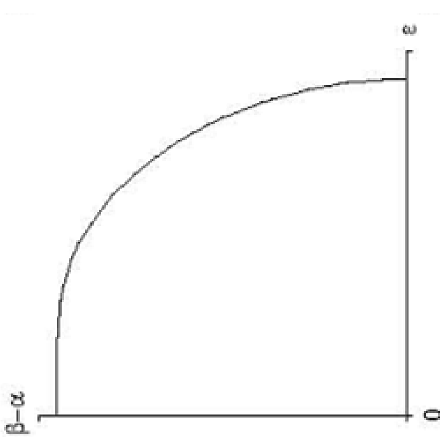
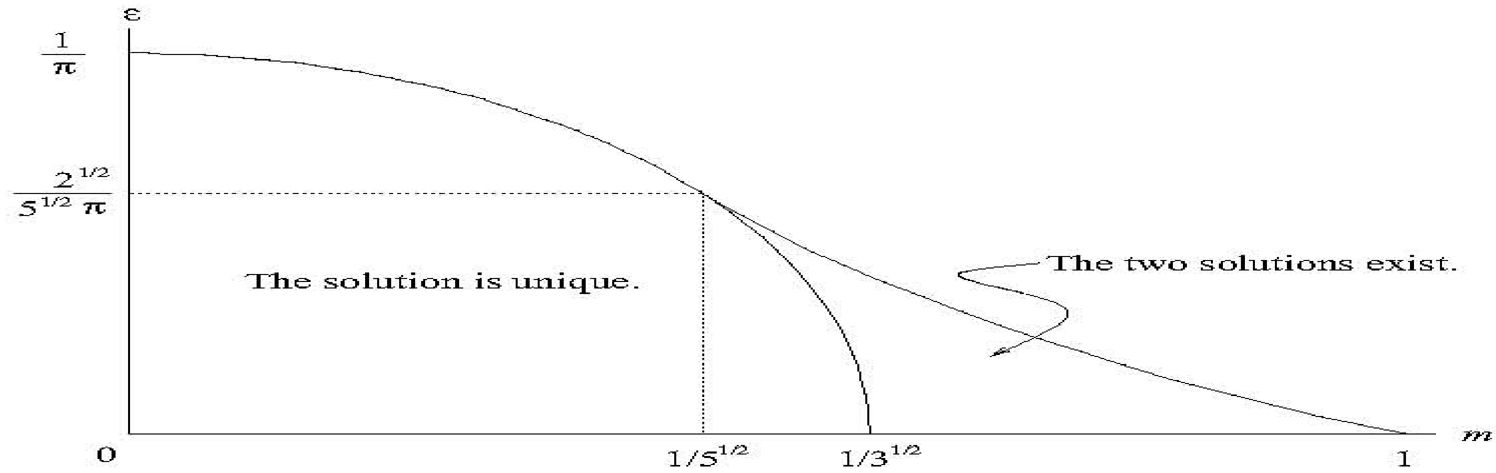
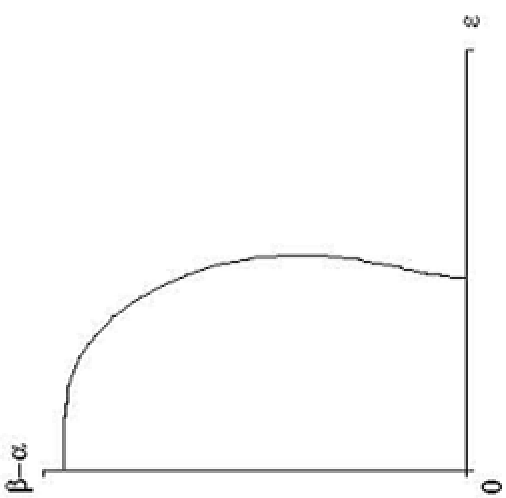
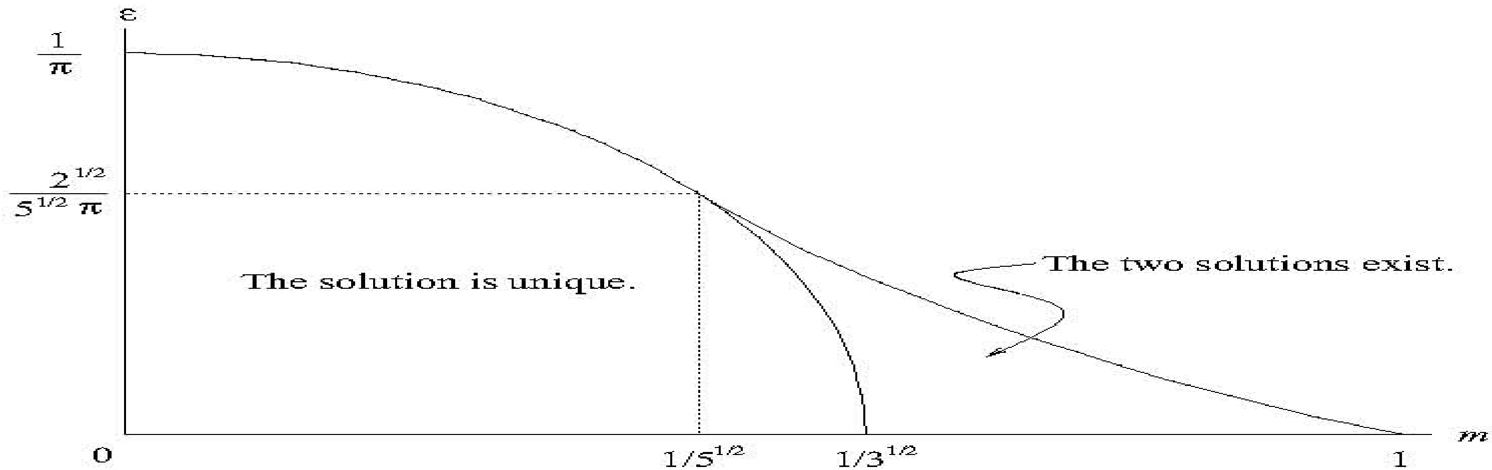
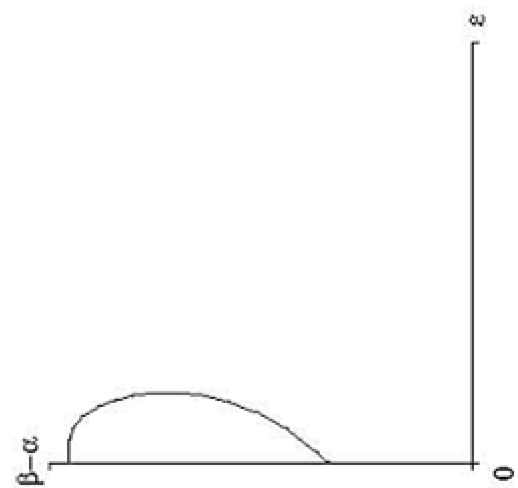
The two solutions exist.
0
1/5^{1/2}
1/3^{1/2}
1
m

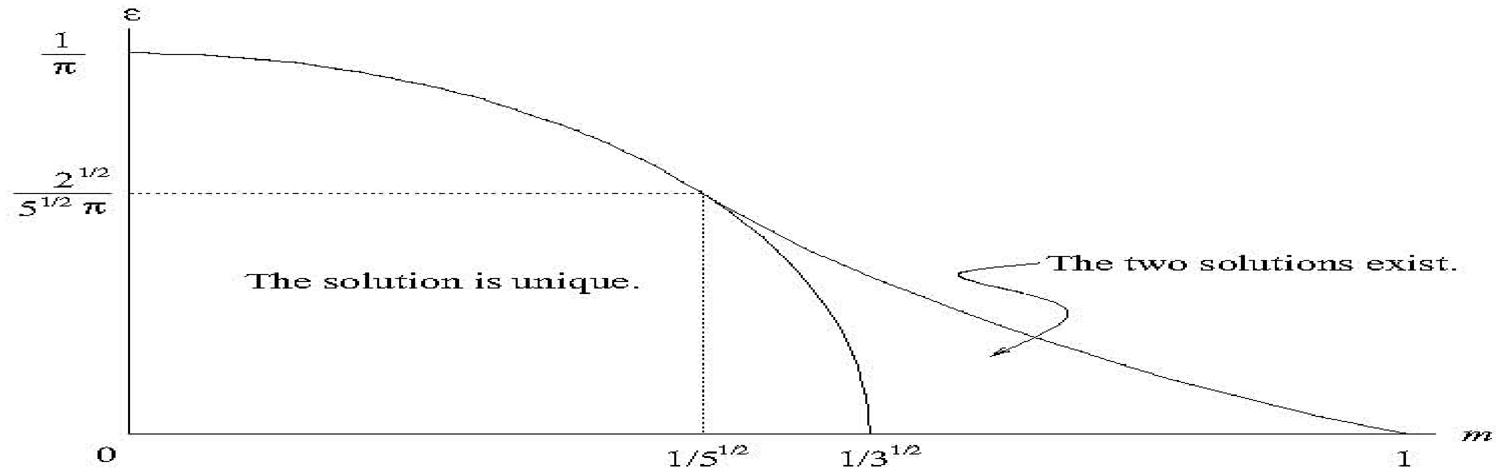

ε
β−α
0

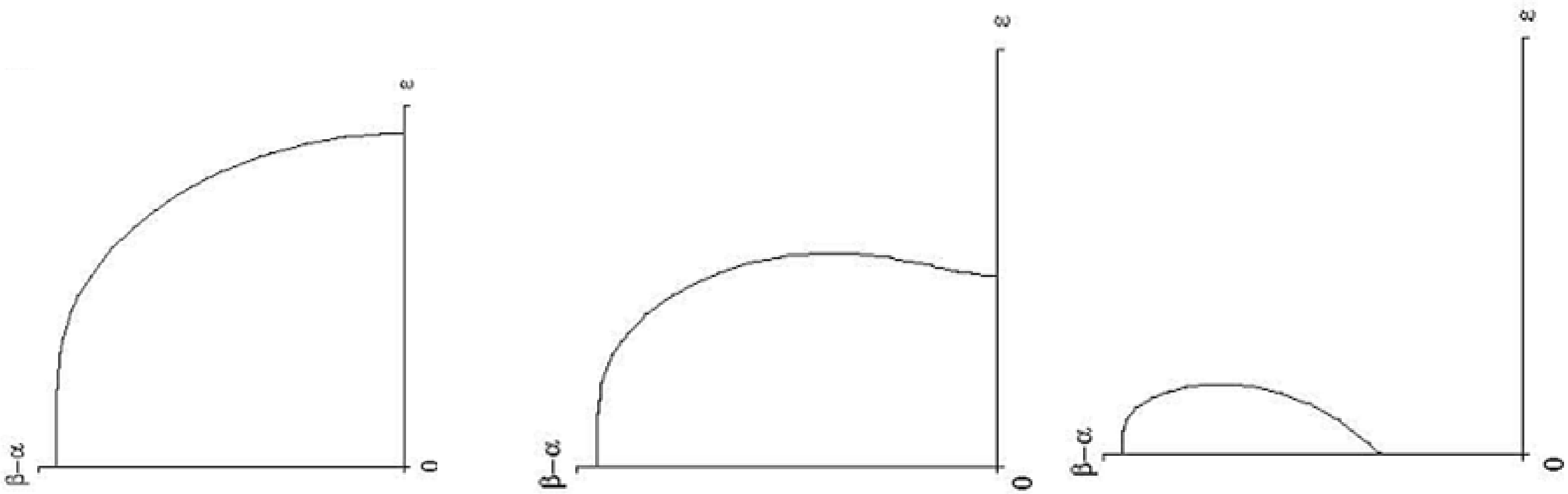

ε
1/π
$2^{1/2}/(5^{1/2}\pi)$
The solution is unique.
The two solutions exist.
0
$1/5^{1/2}$
$1/3^{1/2}$
1
m


ε
β−α
0


ε
$\frac{1}{\pi}$
$\frac{2^{1/2}}{5^{1/2}\pi}$
The solution is unique.
The two solutions exist.
0
$1/5^{1/2}$
$1/3^{1/2}$
1
m


ε
β−α
0


ε
$\frac{1}{\pi}$
$\frac{2^{1/2}}{5^{1/2}\pi}$
The solution is unique.
The two solutions exist.
0
$1/5^{1/2}$
$1/3^{1/2}$
1
m


ε
β−α
0
ε
β−α
0
ε
β−α
0

$$
\begin{cases}
\varepsilon^2 \dfrac{d^2 u}{dx^2}(x) + f(u(x)) - a = 0, & x \in (0,1), \\
\dfrac{du}{dx}(x) = 0, & x = 0, 1, \\
m = \displaystyle\int_0^1 u(x)\,dx, \quad a = \int_0^1 f(u(x))\,dx, &
\end{cases}
\tag{1}
$$

where ε is a positive parameter and f is the cubic polynomial

$$
f(u) := u - u^3.
$$

$$\begin{cases} \varepsilon^2 \dfrac{d^2u}{dx^2}(x) + f(u(x)) - a = 0, & x \in (0,1), \\ \dfrac{du}{dx}(x) = 0, & x = 0, 1, \\ m = \displaystyle\int_0^1 u(x)\,dx, \quad a = \int_0^1 f(u(x))\,dx, \end{cases} \tag{1}$$

where ε is a positive parameter and f is the cubic polynomial

$$f(u) := u - u^3.$$

Proposition 1. *Given $\varepsilon > 0$ and $m \geq 0$, the non-local equation (1) has a monotone increasing solution $u(x)$ if and only if (h, s) satisfies the equation*

$$\varepsilon = \mathcal{E}(h,s), \quad m = \mathcal{M}(h,s), \qquad (h,s) \in (0,1) \times (0,1) \subset \mathbb{R}^2, \tag{2}$$

where

$$\mathcal{E}(h,s) := \frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}, \tag{3}$$

$$\mathcal{M}(h,s) := \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh, \sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}. \tag{4}$$

For the solution (h, s) the equation (1) has a monotone increasing solution in the form

$$u(x) = \frac{\beta(h,s)(1-hs)\,\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \alpha(h,s)\,\mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}{(1-hs)\,\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}, \tag{5}$$

where α and β are defined by

$$\alpha(h,s) := \frac{3hs^2 - 2(1+h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}},$$

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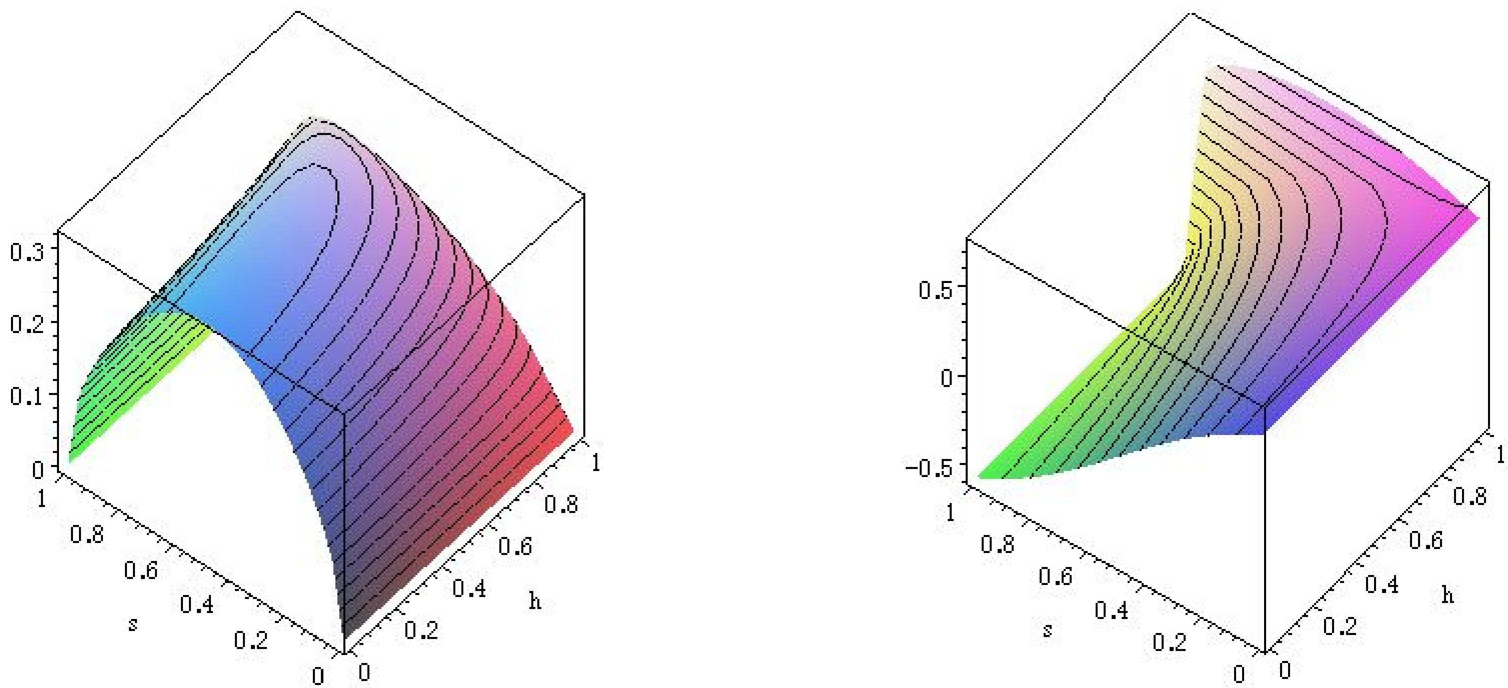

Figure 1 shows the region of (m, ε) where (2) has solutions.

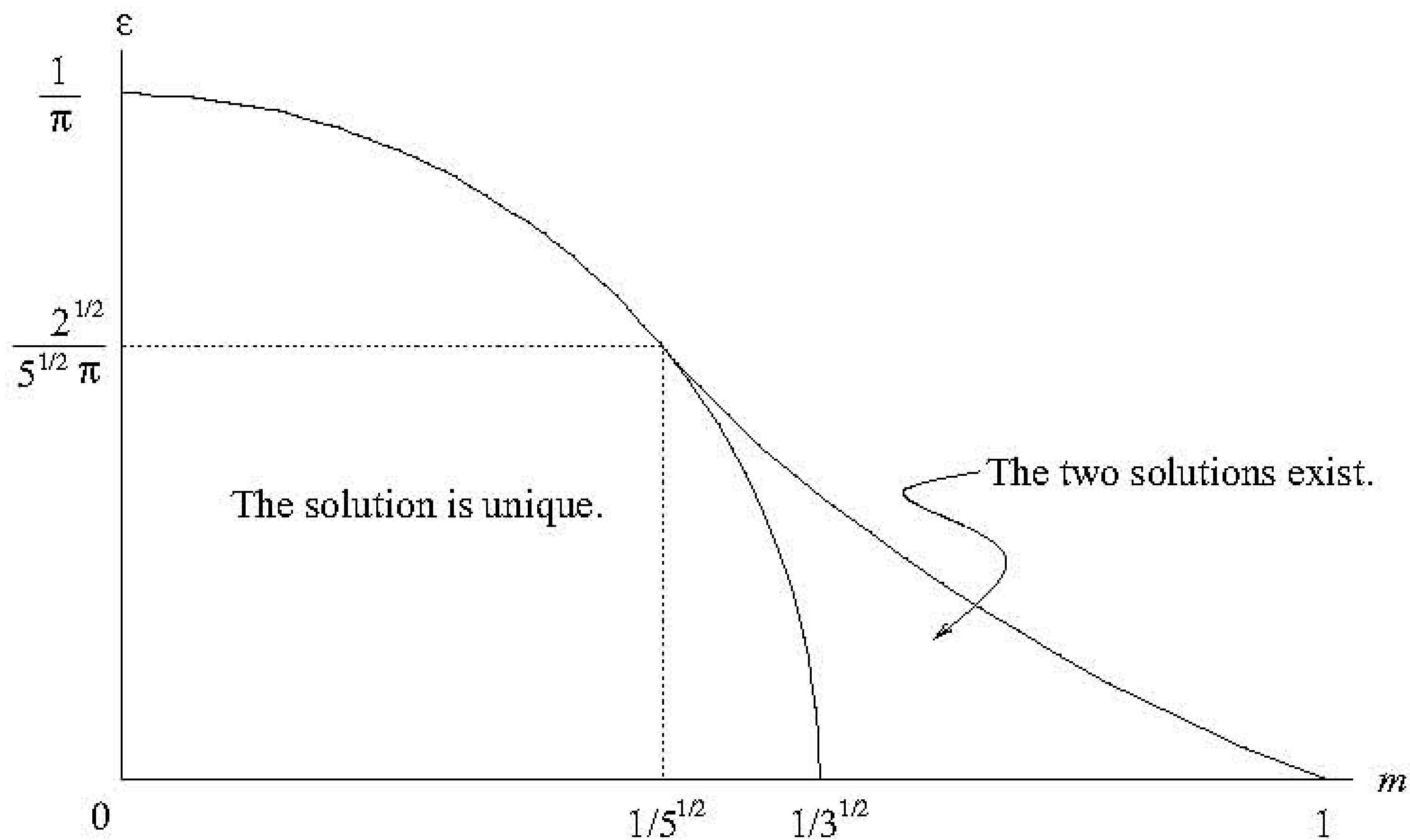

FIGURE 1. The region where (2) has solutions.

Proposition 1. *Given $\varepsilon > 0$ and $m \geq 0$, the non-local equation (1) has a monotone increasing solution $u(x)$ if and only if (h, s) satisfies the equation*

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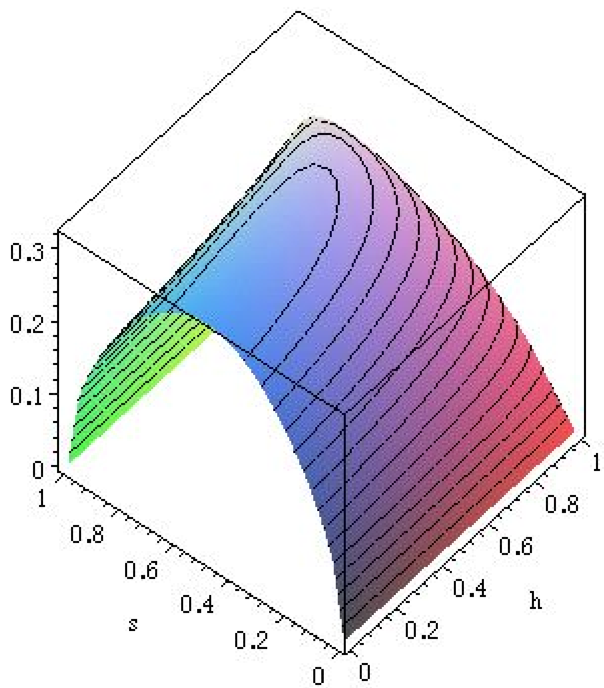
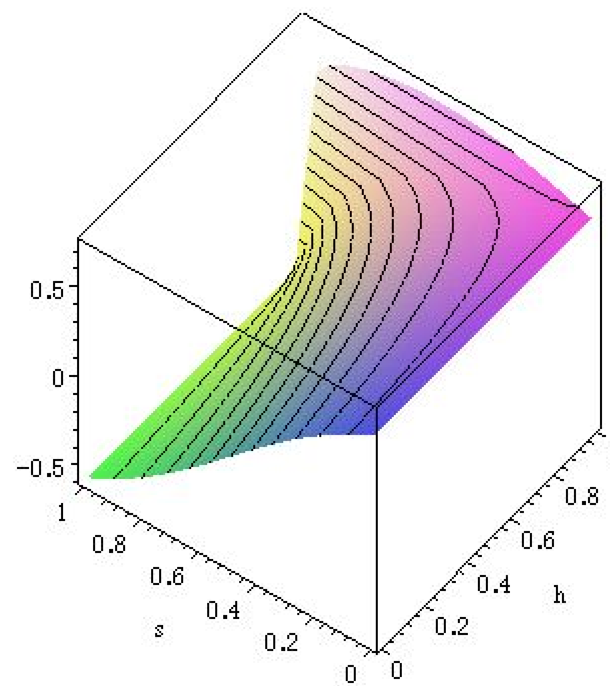




Fix ε= 1/4.

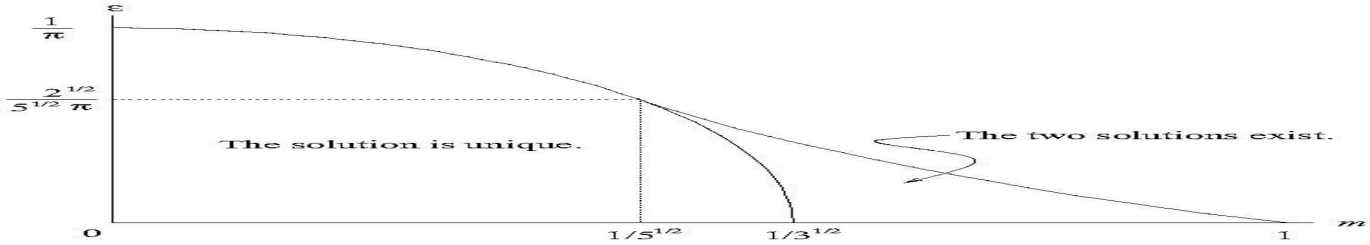
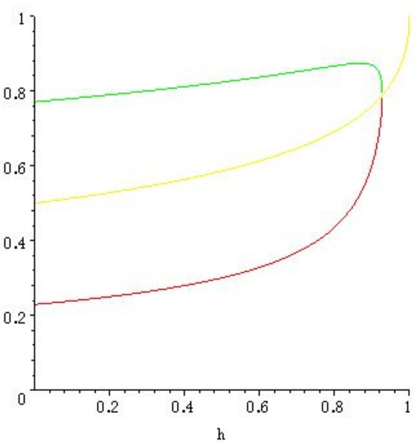


FIGURE 1. The region where (2) has solutions.



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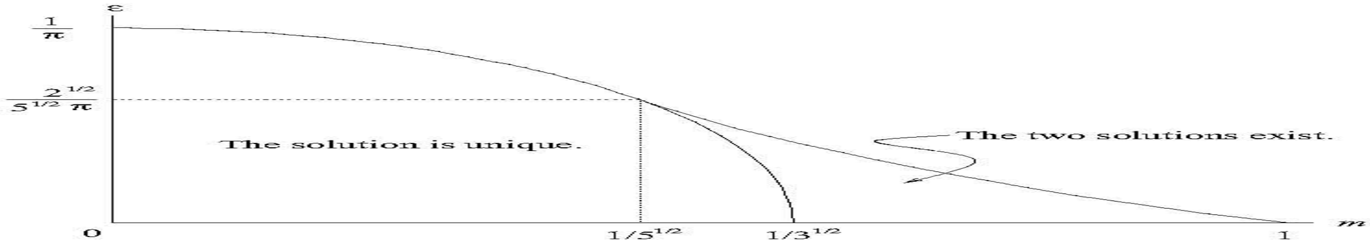

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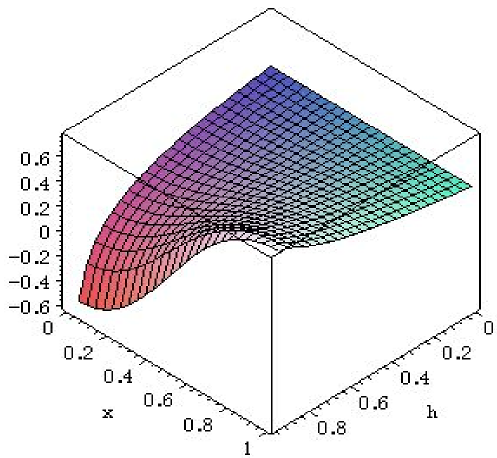

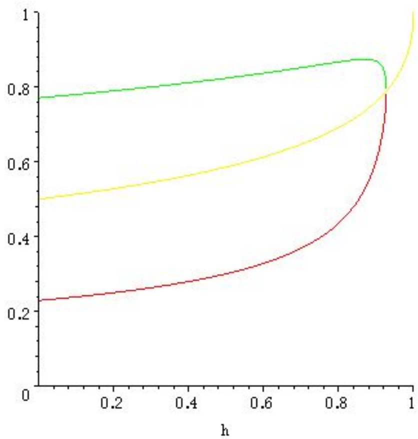

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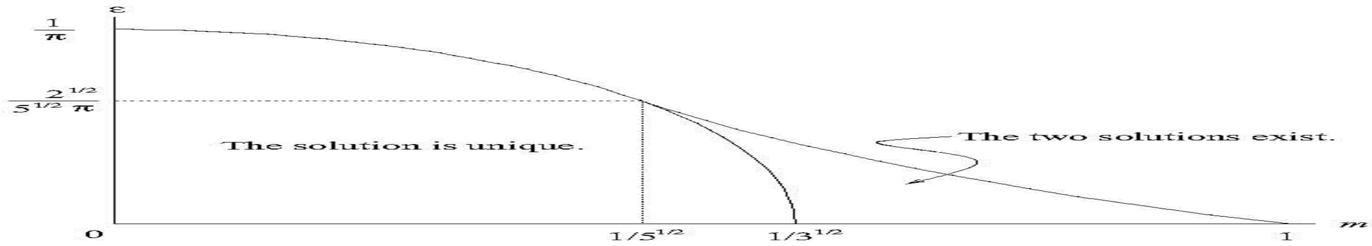

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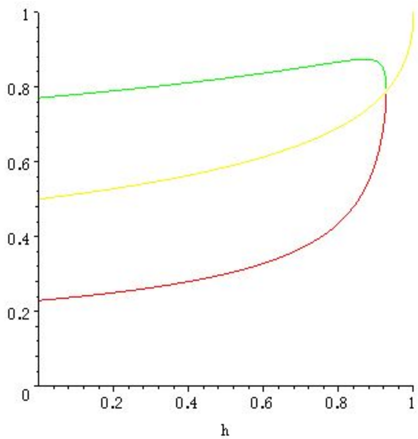

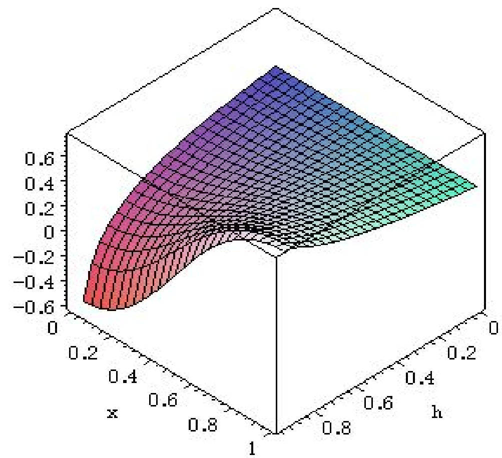

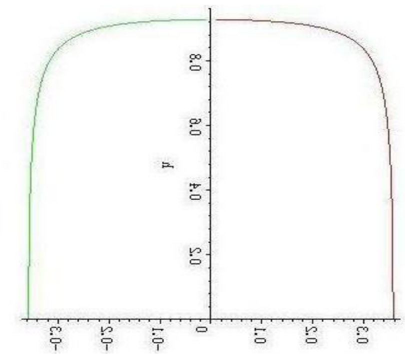

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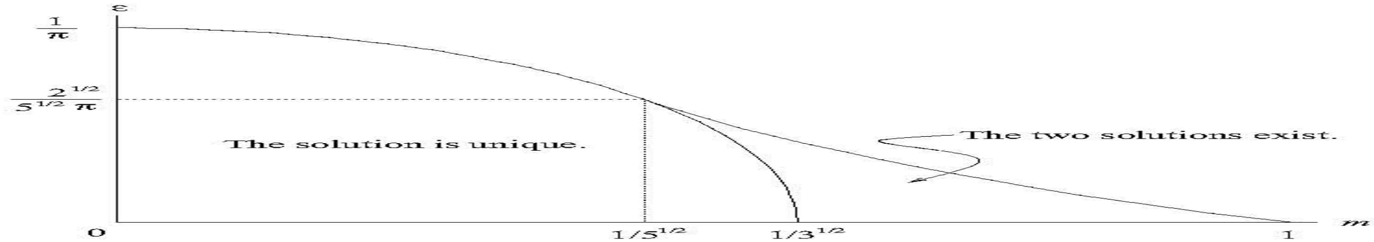

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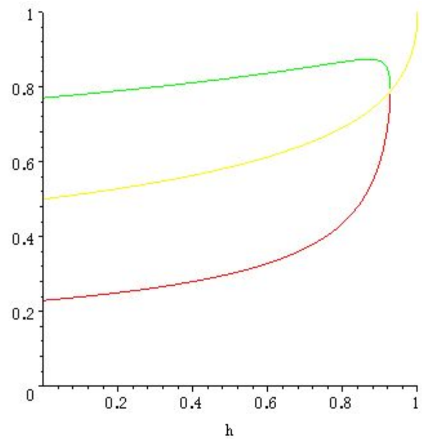
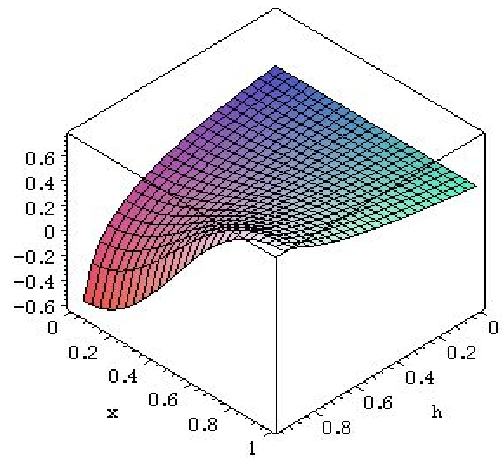
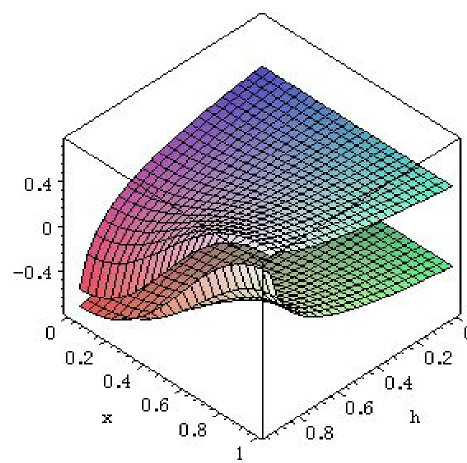
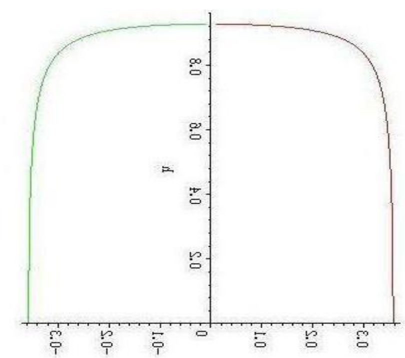
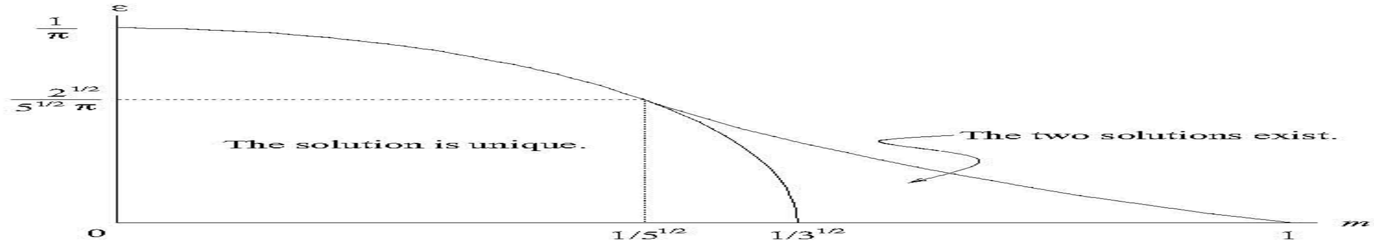








Fix ε= 1/6.



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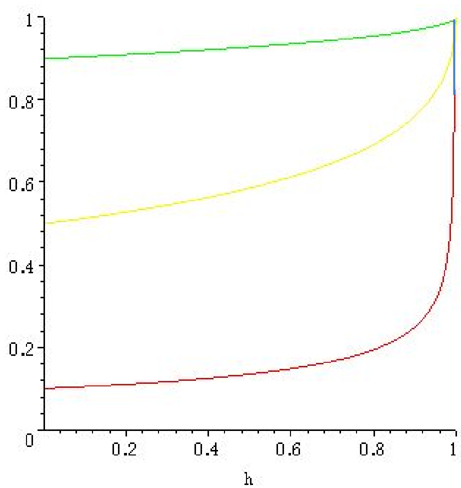

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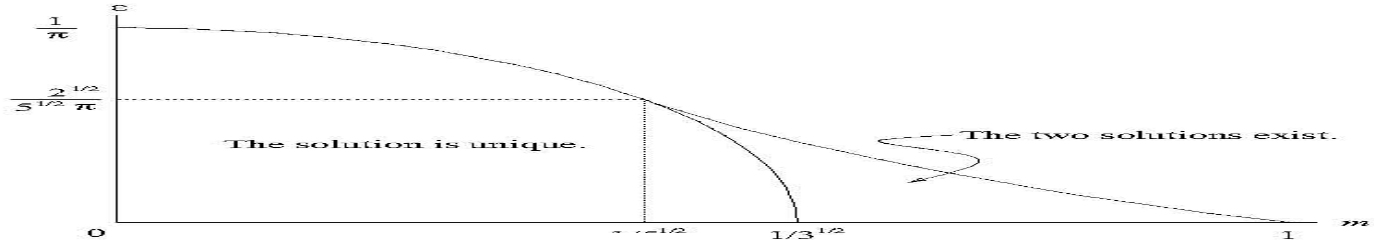
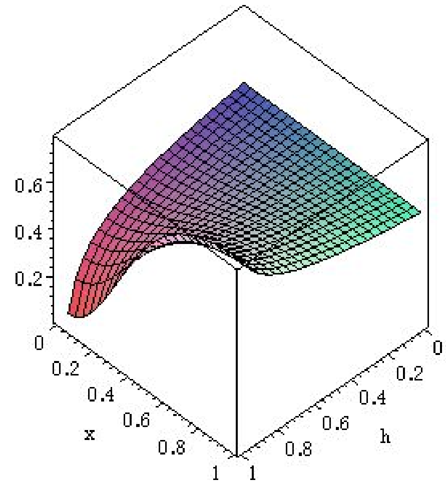
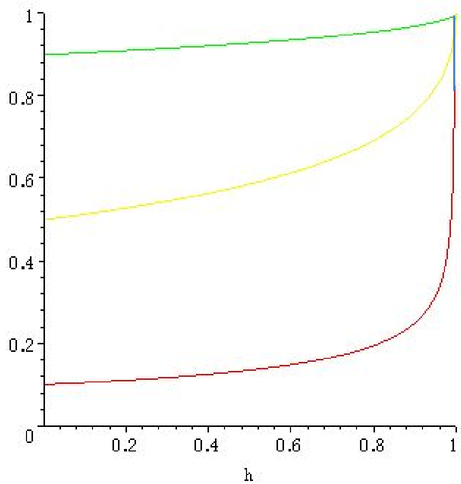
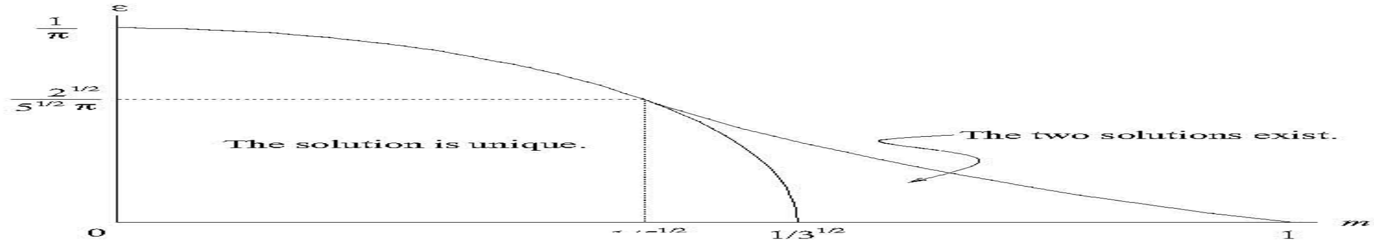


F here (2) has solutions.





Fix ε= 1/6.



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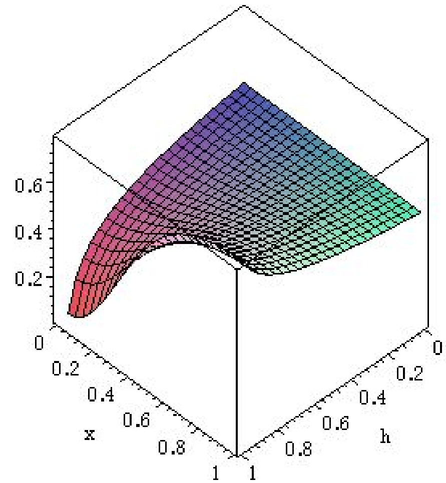

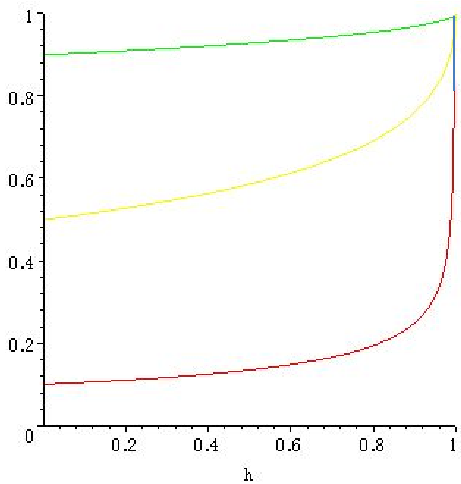

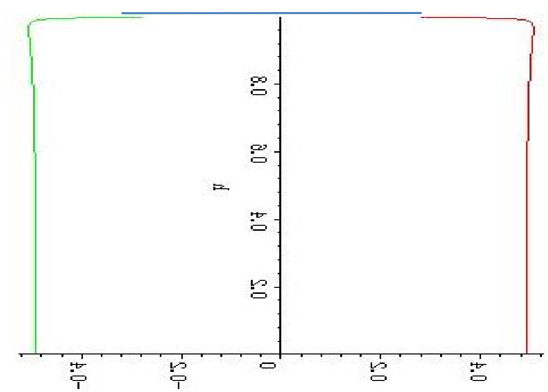

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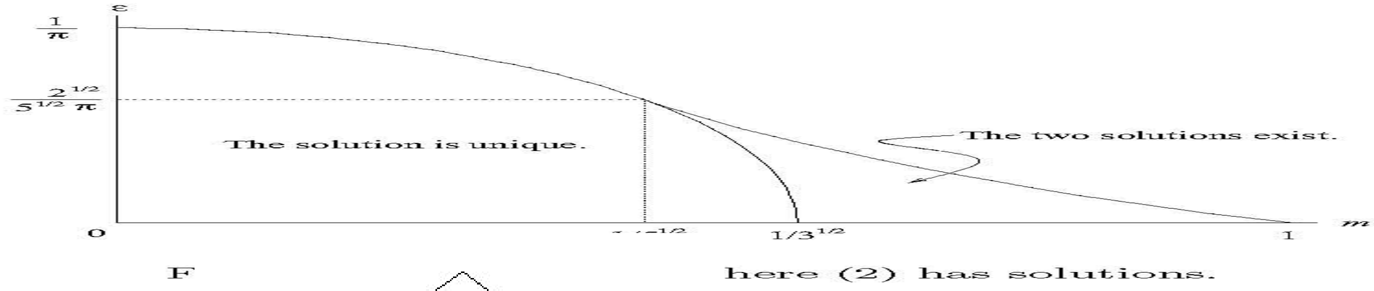
ε
1/π
2^{1/2}/(5^{1/2} π)
The solution is unique.
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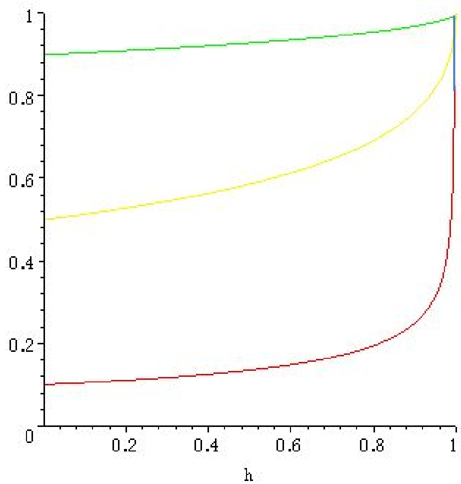
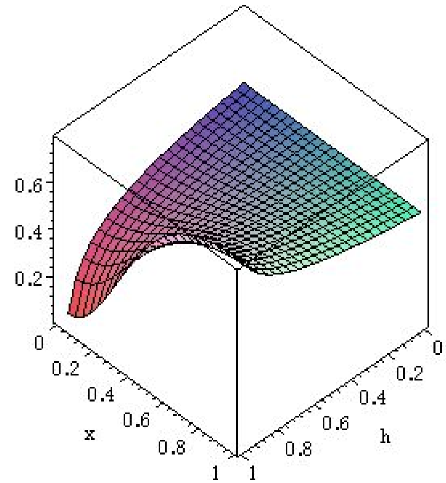
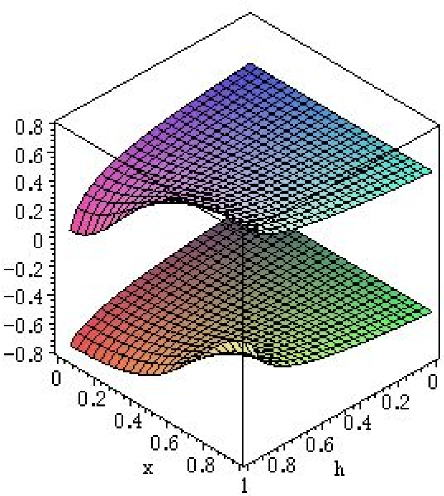
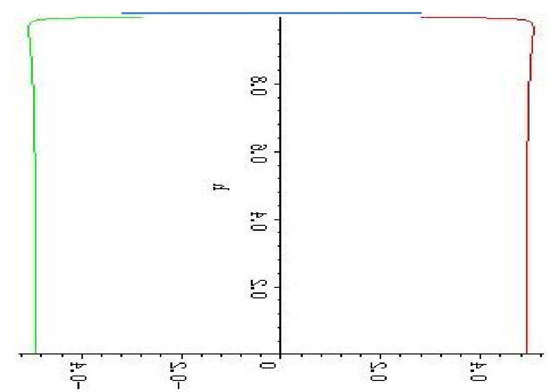
0
1/3^{1/2}
1
m
F
here (2) has solutions.


1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
h


0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
x
h


0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
x
h



Analyzing (3) and (4), we can prove the following result which gives the number of the solutions to (2).

THEOREM 1.1. *Put*

$$\eta_0(\xi) := \begin{cases} \sqrt{1-3\xi^2}/\pi, & \xi \in [0, 1/\sqrt{3}), \\ 0, & \xi \in [1/\sqrt{3}, 1]. \end{cases} \tag{6}$$

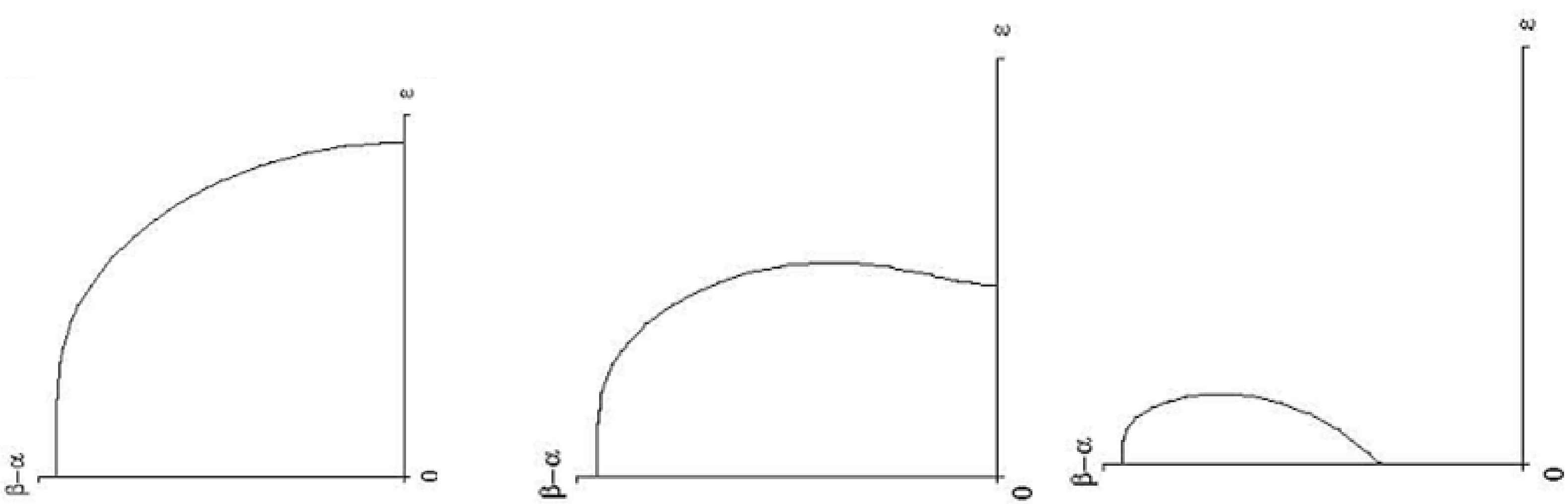
There exists a monotone decreasing continuous function $\eta : [1/\sqrt{5}, 1] \ni \xi \to \eta(\xi) \in [\sqrt{2}/\sqrt{5}\pi, 1/\pi]$ *satisfying* $\eta(\xi) = \eta_0(\xi)$ *at* $\xi \in \{1/\sqrt{5}, 1\}$ *and* $\eta(\xi) > \eta_0(\xi)$ *for* $\xi \in (1/\sqrt{5}, 1)$ *such that solutions* (h, s) *of* (2) *exist if and only if*

$$\begin{aligned}(m,\varepsilon) \in &\{(m,\varepsilon) : 0 < \varepsilon < \eta_0(m),\ 0 \le m \le 1/\sqrt{5}\} \\ &\cup \{(m,\varepsilon) : 0 < \varepsilon \le \eta(m),\ 1/\sqrt{5} < m < 1\}.\end{aligned}$$

The number of the solutions depends on (m,ε) *as follows. If* $\varepsilon \in (0, \eta_0(m))$ *for each* $m \in [0, 1/\sqrt{3}]$ *the solution is unique. If* $\varepsilon \in (\eta_0(m), \eta(m))$ *for each* $m \in (1/\sqrt{5}, 1)$ *the equation* (2) *has two solutions. If* $\varepsilon = \eta_0(m)$ *for each* $m \in (1/\sqrt{5}, 1/\sqrt{3})$ *or* $\varepsilon = \eta(m)$ *for each* $m \in (1/\sqrt{5}, 1)$ *the solution is also unique.*

Proposition 1.2. *Let* $(h, s) = (h(m,\varepsilon), s(m,\varepsilon))$ *be the unique solution to* (2) *for* $\varepsilon \in (0, \sqrt{1-3m^2}/\pi)$ *and* $m \in [0, 1/\sqrt{3})$. *Then*

$$\lim_{\varepsilon\downarrow 0} h(m,\varepsilon) = 1, \qquad \lim_{\varepsilon\downarrow 0} s(m,\varepsilon) = 1,$$
$$\lim_{\varepsilon\downarrow 0} \alpha(h(m,\varepsilon), s(m,\varepsilon)) = -1, \qquad \lim_{\varepsilon\downarrow 0} \beta(h(m,\varepsilon), s(m,\varepsilon)) = 1. \tag{7}$$



$$
\begin{cases}
\varepsilon^2 \dfrac{d^2u}{dx^2}(x) + f(u(x)) - a = 0, & x \in (0,1), \\
\dfrac{du}{dx}(x) = 0, & x = 0, 1, \\
m = \displaystyle\int_0^1 u(x)\,dx, \quad a = \int_0^1 f(u(x))\,dx,
\end{cases} \tag{1}
$$

where ε is a positive parameter and f is the cubic polynomial

$$
f(u) := u - u^3.
$$

Proposition 1.1. *Given $\varepsilon > 0$ and $m \geq 0$, the non-local equation (1) has a monotone increasing solution $u(x)$ if and only if (h, s) satisfies the equation*

$$
\varepsilon = \mathcal{E}(h,s), \quad m = \mathcal{M}(h,s), \qquad (h,s) \in (0,1) \times (0,1) \subset \mathbb{R}^2 \tag{2}
$$

where

$$
\mathcal{E}(h,s) := \frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}, \tag{3}
$$

$$
\mathcal{M}(h,s) := \frac{-(hs^2 - 2(1+h)s + 3) + 4(1-s)(1-sh)\Pi(-sh, \sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(1+h)s + 3}}. \tag{4}
$$

For the solution (h, s) the equation (1) has a monotone increasing solution in the form

$$
u(x) = \frac{\beta(h,s)(1-hs)\,\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \alpha(h,s)\,\mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})}{(1-hs)\,\mathrm{sn}^2(K(\sqrt{h})x, \sqrt{h}) + \mathrm{cn}^2(K(\sqrt{h})x, \sqrt{h})} \tag{5}
$$

$$
a = \frac{2(hs^2 - 2sh + 1)(hs^2 - 2s + 1)(1 - hs^2)}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}^{\,3}}
$$

$$\begin{cases} \varepsilon^2 \dfrac{d^2u}{dx^2}(x) + f(u(x)) - a = 0, & x \in (0,1), \\ \dfrac{du}{dx}(x) = 0, & x = 0, 1, \\ m = \displaystyle\int_0^1 u(x)\,dx, \quad a = \int_0^1 f(u(x))\,dx, \end{cases} \tag{1}$$

where ε is a positive parameter and f is the cubic polynomial

$$f(u) := u - u^3.$$

9.3 Derivation of exact solutions

Assume that $u = u(x)$ is a monotone increasing solution to (1). Multiplying the differential equation in (1) by du/dx and calculating integration, we have

$$\left(\frac{du}{dx}\right)^2 = \frac{F(u)}{2\varepsilon^2}, \quad F(u) := u^4 - 2u^2 + 4au + p \tag{12}$$

for some constant $p \in \mathbb{R}$.

Let $\alpha := u(0)$ and $\beta := u(1)$. It then follows from

$$du/dx(0) = du/dx(1) = 0$$

that

$$F(\alpha) = F(\beta) = 0.$$

Thus

$$F(u) = (u - \alpha)(u - \beta)F_1(u),$$

where $F_1(u)$ is a quadratic polynomial.

$$F(u) := u^4 - 2u^2 + 4au + p$$

Thus

$$F(u) = (u - \alpha)(u - \beta)F_1(u),$$

where $F_1(u)$ is a quadratic polynomial.

Using $F(u(x)) > 0$ and the continuity of $u(x)$ for x, we have

$$F_1(u) < 0 \text{ for all } u \in (\alpha, \beta).$$

Hence there exist γ and δ such that

$$F_1(\gamma) = F_1(\delta) = 0 \text{ and } \gamma \leq \alpha < \beta \leq \delta.$$

Consequently,

$$F(u) = (u - \alpha)(u - \beta)(u - \gamma)(u - \delta), \quad \gamma \leq \alpha < \beta \leq \delta$$

and

$$\alpha + \beta + \gamma + \delta = 0, \quad \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma\delta = -2,$$
$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\delta\gamma + \beta\delta\gamma = -4a, \quad \alpha\beta\gamma\delta = p.$$

Since

$$\frac{du}{dx} = \frac{\sqrt{F(u)}}{\sqrt{2}\varepsilon}$$

we have

$$x = \int_{\alpha}^{u(x)} \frac{\sqrt{2}\varepsilon}{\sqrt{(u-\alpha)(u-\beta)(u-\gamma)(u-\delta)}} du.$$

Here we notice $\gamma < \alpha < \beta < \delta$ by the fact that the integral must be finite.

Changing the variable

$$u = \frac{\gamma(\beta-\alpha)\tau^2 - \alpha(\beta-\gamma)}{(\beta-\alpha)\tau^2 - (\beta-\gamma)}, \quad \tau \in [0,1] \tag{13}$$

yields

$$x = \frac{2\sqrt{2}\varepsilon}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} \int_0^{\sqrt{\frac{(\beta-\gamma)(u(x)-\alpha)}{(\beta-\alpha)(u(x)-\gamma)}}} \frac{1}{\sqrt{1-k^2\tau^2}\sqrt{1-\tau^2}} d\tau,$$

where

$$k = \sqrt{\frac{(\beta-\alpha)(\delta-\gamma)}{(\beta-\gamma)(\delta-\alpha)}}. \tag{14}$$

$$x = \frac{2\sqrt{2}\varepsilon}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} \int_0^{\sqrt{\frac{(\beta-\gamma)(u(x)-\alpha)}{(\beta-\alpha)(u(x)-\gamma)}}} \frac{1}{\sqrt{1-k^2\tau^2}\sqrt{1-\tau^2}}\, d\tau,$$

where

$$k = \sqrt{\frac{(\beta-\alpha)(\delta-\gamma)}{(\beta-\gamma)(\delta-\alpha)}}. \tag{14}$$

Taking $u(1) = \beta$ into account, we obtain

$$1 = \frac{2\sqrt{2}\varepsilon}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} K(k).$$

Thus, we get

$$K(k)x = \int_0^{\sqrt{\frac{(\beta-\gamma)(u(x)-\alpha)}{(\beta-\alpha)(u(x)-\gamma)}}} \frac{1}{\sqrt{1-k^2\tau^2}\sqrt{1-\tau^2}}\, d\tau,$$

which implies

$$K(k)x = \operatorname{sn}^{-1}\left(\sqrt{\frac{(\beta-\gamma)(u(x)-\alpha)}{(\beta-\alpha)(u(x)-\gamma)}}, k\right)$$

Hence, we have

$$\operatorname{sn}(K(k)x, k) = \sqrt{\frac{(\beta-\gamma)(u(x)-\alpha)}{(\beta-\alpha)(u(x)-\gamma)}}.$$

Hence, we have

$$\operatorname{sn}(K(k)x, k) = \sqrt{\frac{(\beta - \gamma)(u(x) - \alpha)}{(\beta - \alpha)(u(x) - \gamma)}}.$$

Hence, we have

$$\operatorname{sn}(K(k)x, k) = \sqrt{\frac{(\beta - \gamma)(u(x) - \alpha)}{(\beta - \alpha)(u(x) - \gamma)}}.$$

Therefore, we obtain

$$u(x) = \frac{\gamma(\beta - \alpha)\operatorname{sn}^2(K(k)x, k) - \alpha(\beta - \gamma)}{(\beta - \alpha)sn^2(K(k)x, k) - (\beta - \gamma)}. \tag{15}$$

The integral (1) are also expressed by the Jacobi elliptic integrals with α, β, γ, and δ. Substituting (13) into it leads us to

$$\begin{aligned} m = \int_0^1 u(x)\,dx &= \int_\alpha^\beta \frac{\sqrt{2}\varepsilon u}{\sqrt{F(u)}}du \\ &= \sqrt{2}\varepsilon\left\{\int_\alpha^\beta \frac{u-\gamma}{\sqrt{F(u)}}du + \int_\alpha^\beta \frac{\gamma}{\sqrt{F(u)}}du\right\} \\ &= \sqrt{2}\varepsilon\left\{\frac{2(\alpha-\gamma)\Pi(\nu,k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} + \frac{2\gamma K(k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}}\right\}, \end{aligned}$$

where $\nu = -(\beta-\alpha)/(\beta-\gamma)$ and k is the same as in (14).

In consequence, if u is a monotone increasing solution to (1), then it is written in the form (15), where α, β, γ, δ, k, ν, ε, m, a, and p satisfy

$$\begin{aligned} &\gamma < \alpha < \beta < \delta, \\ &\alpha+\beta+\gamma+\delta = 0, \quad \alpha(\beta+\gamma+\delta)+\beta(\gamma+\delta)+\gamma\delta = -2, \\ &a = -(\alpha\beta\gamma+\alpha\beta\delta+\alpha\delta\gamma+\beta\delta\gamma)/4, \quad p = \alpha\beta\gamma\delta, \\ &k = \sqrt{(\beta-\alpha)(\delta-\gamma)/(\beta-\gamma)(\delta-\alpha)}, \quad \nu = -(\beta-\alpha)/(\beta-\gamma), \\ &\varepsilon = \sqrt{(\delta-\alpha)(\beta-\gamma)}/2\sqrt{2}K(k), \\ &m = \sqrt{2}\varepsilon\left\{\frac{2(\alpha-\gamma)\Pi(\nu,k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} + \frac{2\gamma K(k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}}\right\}. \end{aligned}$$

$$\gamma < \alpha < \beta < \delta,$$

$$\alpha + \beta + \gamma + \delta = 0, \quad \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma\delta = -2,$$

$$k = \sqrt{(\beta - \alpha)(\delta - \gamma)/(\beta - \gamma)(\delta - \alpha)}, \quad \nu = -(\beta - \alpha)/(\beta - \gamma),$$

$$\gamma < \alpha < \beta < \delta,$$

$$\alpha + \beta + \gamma + \delta = 0, \quad \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma\delta = -2,$$

$$k = \sqrt{(\beta - \alpha)(\delta - \gamma)/(\beta - \gamma)(\delta - \alpha)}, \quad \nu = -(\beta - \alpha)/(\beta - \gamma),$$

Put $h = k^2$ and $s = -\nu/k^2$. Then the above relations allow the following explicit expressions with respect to (h, s):

$$\gamma < \alpha < \beta < \delta,$$

$$\alpha + \beta + \gamma + \delta = 0, \quad \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma\delta = -2,$$

$$k = \sqrt{(\beta - \alpha)(\delta - \gamma)/(\beta - \gamma)(\delta - \alpha)}, \quad \nu = -(\beta - \alpha)/(\beta - \gamma),$$

Put $h = k^2$ and $s = -\nu/k^2$. Then the above relations allow the following explicit expressions with respect to (h, s):

$$\alpha = \frac{3hs^2 - 2(1+h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \tag{16}$$

$$\beta = \frac{-hs^2 - 2(1-h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \tag{17}$$

$$\gamma = \frac{-hs^2 + 2(1+h)s - 3}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \tag{18}$$

$$\delta = \frac{-hs^2 + 2(1-h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \tag{19}$$

$$a = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\delta\gamma + \beta\delta\gamma)/4, \quad p = \alpha\beta\gamma\delta,$$

$$\varepsilon = \sqrt{(\delta-\alpha)(\beta-\gamma)}/2\sqrt{2}K(k),$$

$$m = \sqrt{2}\varepsilon \left\{ \frac{2(\alpha-\gamma)\Pi(\nu,k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} + \frac{2\gamma K(k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}} \right\}.$$

Conversely, if there exists a solution (h, s) to (2), then it follows from a straight calculation that (15) is a solution to (1).

$$a = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\delta\gamma + \beta\delta\gamma)/4, \quad p = \alpha\beta\gamma\delta,$$

$$\varepsilon = \sqrt{(\delta - \alpha)(\beta - \gamma)}/2\sqrt{2}K(k),$$

$$m = \sqrt{2}\varepsilon \left\{ \frac{2(\alpha - \gamma)\Pi(\nu, k)}{\sqrt{(\delta - \alpha)(\beta - \gamma)}} + \frac{2\gamma K(k)}{\sqrt{(\delta - \alpha)(\beta - \gamma)}} \right\}.$$

$$\alpha = \frac{3hs^2 - 2(1+h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \tag{16}$$

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$$\delta = \frac{-hs^2 + 2(1-h)s + 1}{\sqrt{3h^2s^4 - 4(h^2+h)s^3 + (4h^2+2h+4)s^2 - 4(h+1)s + 3}}, \tag{19}$$

Conversely, if there exists a solution (h, s) to (2), then it follows from a straight calculation that (15) is a solution to (1).

$$a=-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\delta\gamma+\beta\delta\gamma)/4, \quad p=\alpha\beta\gamma\delta,$$
$$\varepsilon=\sqrt{(\delta-\alpha)(\beta-\gamma)}/2\sqrt{2}K(k),$$
$$m=\sqrt{2}\varepsilon\left\{\frac{2(\alpha-\gamma)\Pi(\nu,k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}}+\frac{2\gamma K(k)}{\sqrt{(\delta-\alpha)(\beta-\gamma)}}\right\}.$$

$$\alpha=\frac{3hs^2-2(1+h)s+1}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}}, \tag{16}$$
$$\beta=\frac{-hs^2-2(1-h)s+1}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}}, \tag{17}$$
$$\gamma=\frac{-hs^2+2(1+h)s-3}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}}, \tag{18}$$
$$\delta=\frac{-hs^2+2(1-h)s+1}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}}, \tag{19}$$

.

$$\varepsilon=\frac{\sqrt{2s(1-s)(1-sh)}/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(1+h)s+3}}, \tag{20}$$
$$m=\frac{-(hs^2-2(1+h)s+3)+4(1-s)(1-sh)\Pi(-sh,\sqrt{h})/K(\sqrt{h})}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(1+h)s+3}}, \tag{21}$$
$$a=\frac{2(hs^2-2sh+1)(hs^2-2s+1)(1-hs^2)}{\sqrt{3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3}^{\,3}},$$
$$p=(3hs^2-2(1+h)s+1)(-hs^2-2(1-h)s+1)(hs^2-2(1+h)s+3)$$
$$(hs^2-2(1-h)s-1)(3h^2s^4-4(h^2+h)s^3+(4h^2+2h+4)s^2-4(h+1)s+3)^{-2},$$

and $(h,s)\in(0,1)\times(0,1)$. Thus we obtain (2). Conversely, if there exists a solution (h,s) to (2), then it follows from a straight calculation that (15) is a solution to (1).