保存則を持つ反応拡散方程式系の安定解とその空間的形状 (I) (II)

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偏微分方程式レクチャーシリーズ
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Part I

18日16:00-17:30

1. Introduction

Reaction-diffusion system with mass conservation

$$\begin{cases} u_t = d_1 \Delta u + f(u,v), \\ v_t = d_2 \Delta v - f(u,v), \end{cases} \qquad \Omega \subset \mathrm{R}^N$$

with the Neumann boundary condition.

Mass conservation by $$\frac{d}{dt}\int_\Omega (u(x,t)+v(x,t))dx = 0$$

Specifically, we are interested in the system with the form

$$\begin{cases} u_t = d_1 \Delta u - g(u+\gamma v) + v, \\ v_t = d_2 \Delta v + g(u+\gamma v) - v, \end{cases} \qquad 0 \le \gamma \le 1$$

Specific examples:

Otsuji-Ishihara-et al (2007)
Ishihara-Otsuji-Mochizuki (2007)

$$\gamma = 0, \quad g(u) = \frac{au}{u^2 + b}$$

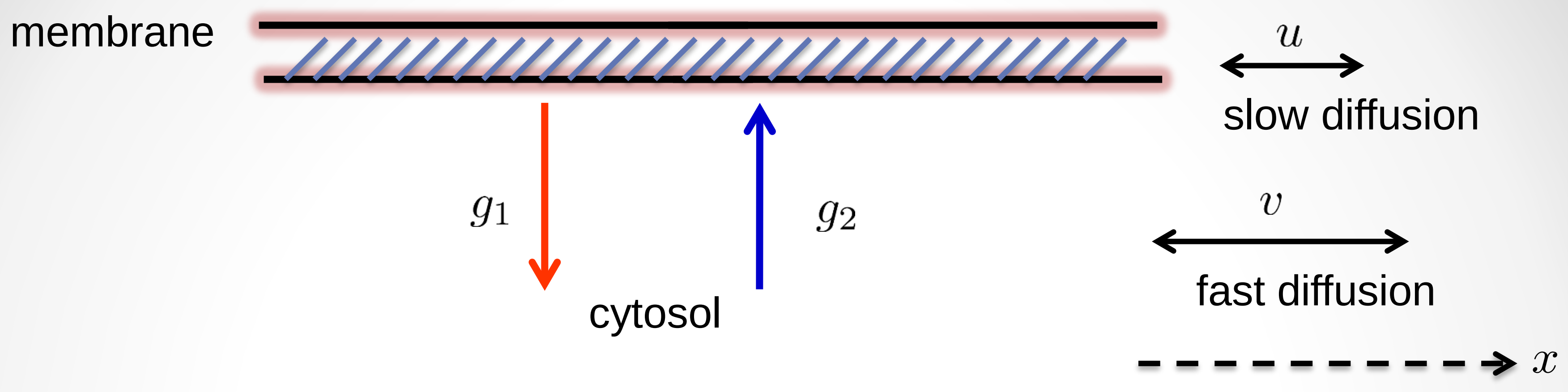
Type I

$$\begin{cases} u_t = d_1 u_{xx} - \dfrac{au}{u^2+b} + v, \\ v_t = d_2 v_{xx} + \dfrac{au}{u^2+b} - v, \end{cases}$$

$$\gamma = 1, \quad g(w) = \frac{w}{(aw+1)^2}$$

Type II

$$\begin{cases} u_t = d_1 u_{xx} - \dfrac{u+v}{(a(u+v)+1)^2} + v, \\ v_t = d_2 v_{xx} + \dfrac{u+v}{(a(u+v)+1)^2} - v, \end{cases}$$



u : the concentration of the activating protein in the membrane

v : the concentration of the inactivating protein in the cytosol

$g_1 = g_1(u, v), \; g_2 = g_2(u, v)$ transport rates

$$\frac{du}{dt} = -g_1 + g_2$$

$$\frac{dv}{dt} = g_1 - g_2$$

$$u_t = d_1 u_{xx} - g_1(u, v) + g_2(u, v)$$

$$v_t = d_2 v_{xx} + g_1(u, v) - g_2(u, v)$$

$$0 < d_1 << d_2$$

Numerical simulations

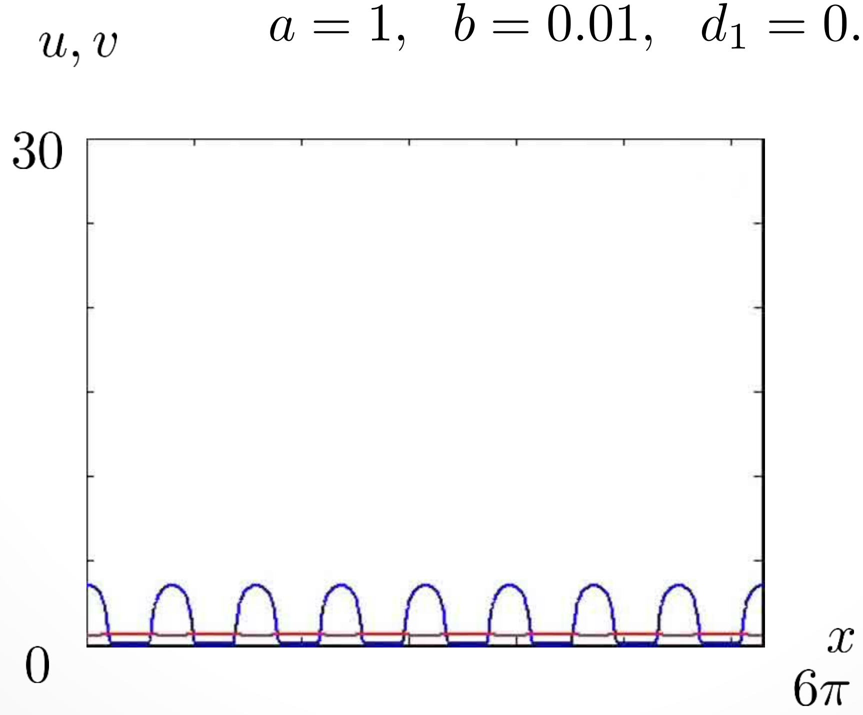
Numerical simulation for the equations with periodic boundary condition

$$\begin{cases} u_t = d_1 u_{xx} - \dfrac{au}{u^2+b} + v, \\ v_t = d_2 v_{xx} + \dfrac{au}{u^2+b} - v, \end{cases}$$

Periodic boundary conditions

$$s_m := \frac{1}{L}\int_0^L (u(x,t)+v(x,t))dx$$

$$a = 1, \quad b = 0.01, \quad d_1 = 0.02, \quad d_2 = 1, \quad s_m = 2$$



Blue: $u(x,t)$

Red: $v(x,t)$

M-Ogawa (2010) (Nonlinearity Vol.23)

``Stability and bifurcation of nonconstant solutions to a reaction-diffusion system with conservation of a mass”

Type I

$$\begin{cases} u_t = d\Delta u - g(u) + v, \\ v_t = \Delta v + g(u) - v \end{cases} \quad \text{in } \ \Omega \ \text{(bounded)} \quad \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega$$

$$m := \frac{1}{|\Omega|} \int_\Omega [u(x,t) + v(x,t)] \, dx \qquad g(u) := \frac{au}{u^2 + b}$$

Turing instability takes place.

We are interested in the localized pattern (spiky pattern)

$$z = du + v \qquad \begin{cases} u_t = d\Delta u - g(u) - du + z, \\ (1-d)u_t + z_t = \Delta z \end{cases}$$

Similar to the Fix-Caginalp phase-field model:

$$\phi_t = d\Delta\phi + \phi - \phi^3 + T$$
$$\alpha\phi_t + \mu T_t = \Delta T$$

Look at the stationary problem:

$$\begin{cases} d\Delta u - g(u) + v = 0, \\ \\ \Delta v + g(u) - v = 0 \end{cases} \qquad m := \frac{1}{|\Omega|}\int_\Omega [u(x) + v(x)]\; dx$$

$$d\Delta u + \Delta v = 0 \qquad du + v = d\langle u\rangle + \langle v\rangle = m - (1-d)\langle u\rangle$$

$$d\Delta u - g(u) - du + m - (1-d)\langle u\rangle = 0, \qquad \langle u\rangle := \frac{1}{|\Omega|}\int_\Omega u(x)\; dx$$

$$E(u) := \int_\Omega \left\{ \frac{d}{2}|\nabla u|^2 + G(u) + \frac{d}{2}u^2 \right\} dx + \frac{|\Omega|}{2(1-d)}\left(m - (1-d)\langle u\rangle\right)^2$$

M (2012)

``Spectrum comparison for a conserved reaction-diffusion system with a variational property'' (J. Applied Analysis and Computation, Vol.2)

Ref.

Bates-Fife (1990),
``Spectral comparison principles for the Cahn-Hilliard and phase-field equations, and time scales for coarsening'' (Physica D)

Fife, ``Models for phase separation and their mathematics'',
Electron. J. Diff. Eqns., Vol. 2000(2000), No. 48, pp. 1-26.

Type II

$$\begin{cases} u_t = d_1 u_{xx} - \dfrac{u+v}{(a(u+v)+1)^2} + v, \\ v_t = d_2 v_{xx} + \dfrac{u+v}{(a(u+v)+1)^2} - v, \end{cases}$$

In general

$$\begin{cases} u_t = d\Delta u - g(u+v) + v, \\ v_t = \Delta v + g(u+v) - v, \end{cases}$$

Jimbo-M (2013, JDE)

Extension of the spectral comparison

(a kind of homotopy or continuation method)

Remark

K. Pham et al (2011), J. Biol. Dynamics

``Density-dependent quiescence in glioma invasion a model for migration/proliferation dichotomy"

$$u_t = -\mu[\Gamma(u+v)u - (1-\Gamma(u+v))v] + r(1-(u+v))u$$
$$v_t = v_{xx} + \mu[\Gamma(u+v)u - (1-\Gamma(u+v))v]$$

u : proliferating population

v : migrating population

μ : changing rate of their phenotype

r : proliferation rate

$\Gamma(w)$:probability from an immotile cell to motile

$r > 0$ Go-on-grow model

$r = 0$ Go-on-rest model

$$u_t = -\mu[g(u+v) - v]$$
$$v_t = v_{xx} + \mu[g(u+v) - v]$$

$$\Gamma(w) = \frac{1}{2}\{1+\tanh(\alpha[\rho^* - w])\}$$

(monotone decreasing)

$$(\Gamma(w) = \frac{1}{2}\{1-\tanh(\alpha[\rho^* - w])\})$$

(monotone increasing)

$$g(w) := \Gamma(w)w$$

Remark: $g(w) = \left(\frac{1}{(w+1)^2}\right)w$

Remark

Latos-M-Suzuki (2018)

$$\begin{cases} u_t = d\Delta u - g(u+v) + v, \\ \tau v_t = \Delta v + g(u+v) - v \end{cases} \qquad \tau \neq 1$$

Jimbo-M (2017)

$$\begin{cases} u_t = d\Delta u - g(u+\gamma v) + v, \\ \tau v_t = \Delta v + g(u+\gamma v) - v, \end{cases} \qquad 0 < \tau, \qquad 0 \leq \gamma \leq 1$$

$$w = u + \gamma v, \qquad z = du + v$$

$$\begin{cases} \alpha w_t - \beta z_t = d\Delta w - (1-d\gamma)g(w) - dw + z, \\ \xi w_t + \delta z_t = \Delta z, \end{cases}$$

$$\alpha = \frac{1-\tau\gamma d^2}{1-d\gamma}, \quad \beta = \frac{\gamma(1-\tau d)}{1-d\gamma}, \quad \xi = \frac{1-\tau d}{1-d\gamma} \quad \delta = \frac{\tau-\gamma}{1-d\gamma}$$

If $\gamma \leq \tau < 1/d$ then $\alpha > 1, \quad \beta \geq 0, \quad \xi > 0, \quad \delta \geq 0$

In general we start from

$$\begin{cases} \alpha w_t - \beta z_t = d\Delta w + f(w) + z, \\ \xi w_t + \delta z_t = \Delta z, \end{cases}$$

$$\alpha > 1, \quad \beta \geq 0, \quad \xi > 0, \quad \delta \geq 0$$

$\beta = 0, \quad f(w) = w(1 - w^2)$ Phase-field system (Fix 1983, Caginalp 1986)

$\delta = 0$

$$\xi w_t = \Delta z, \qquad \xi w_{tt} = \Delta z_t$$

$$-\alpha \Delta w_t + \beta \Delta z_t = -\Delta(d\Delta w + f(w) + z),$$

$$\xi w_t + \beta \xi w_{tt} = -\Delta(d\Delta w + f(w) - \alpha w_t)$$

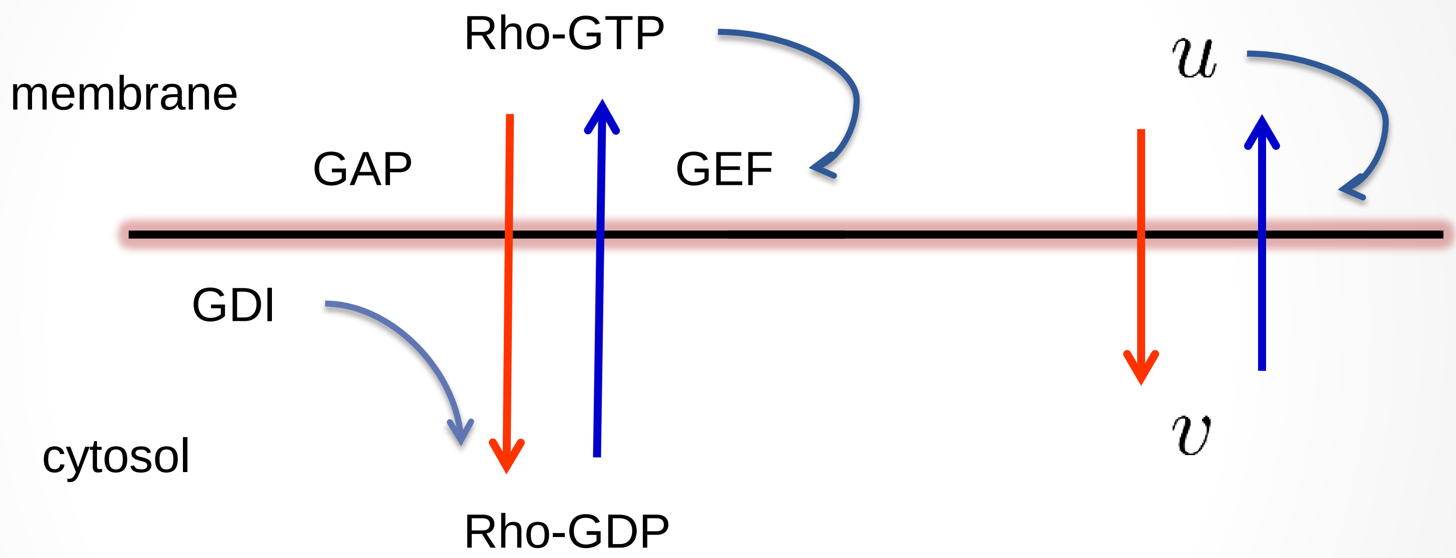
$$\beta = 0, \quad f(w) = w(1 - w^2), \quad \xi = (1 - \alpha)$$

Viscous Cahn-Hilliard equation (A.Novick-Cohen 1988)

Remark: <u>Wave-pining model</u>

Y.Mori, A.Jikkine, and L.Edelstein-Keshet (2008), Biophysical J.

Wave-Pinning and Cell Polarity from a Bistable Reaction-Diffusion System



Model equations:

$$\frac{du}{dt} = f(u,v)$$
$$\frac{dv}{dt} = -f(u,v)$$

$$f(u,v) = \left(k_0 + \frac{\gamma u^2}{K^2+u^2}\right)v - ru$$

A simpler model

$$u_t = \varepsilon^2 u_{xx} + u(u-1)(v+1-u), \qquad 0 < x < 1$$
$$v_t = D v_{xx} - u(u-1)(v+1-u),$$
$$u_x = v_x = 0 \qquad (x = 0,\ 1) \qquad m := \int_0^1 (u+v)dx$$

Stationary problem:

$$\varepsilon^2 u_{xx} + u(u-1)(v+1-u) = 0,$$
$$D v_{xx} - u(u-1)(v+1-u) = 0. \qquad m := \int_0^1 (u+v)dx$$

Constant steady state:

$$(u,v) = (0,m): \quad \text{stable} \qquad (u,v) = (1, m-1): \quad \text{unstable} \quad (m>1)$$

$$(u,v) = \left(\frac{m+1}{2}, \frac{m-1}{2}\right): \quad \text{stable} \quad (m>1)$$

$D \to \infty$

$$\varepsilon^2 u_{xx} + u(u-1)(V+1-u) = 0,$$
$$u_x = 0 \quad (x = 0, 1),$$
$$m = \int_0^1 u(x)\ dx + V.$$

$$\varepsilon^2 u_{xx} + u(u-1)\left(m + 1 - u - \int_0^1 u\ dx\right) = 0,$$
$$u_x = 0 \quad (x = 0, 1).$$

Bifurcation structure of non-constant solutions

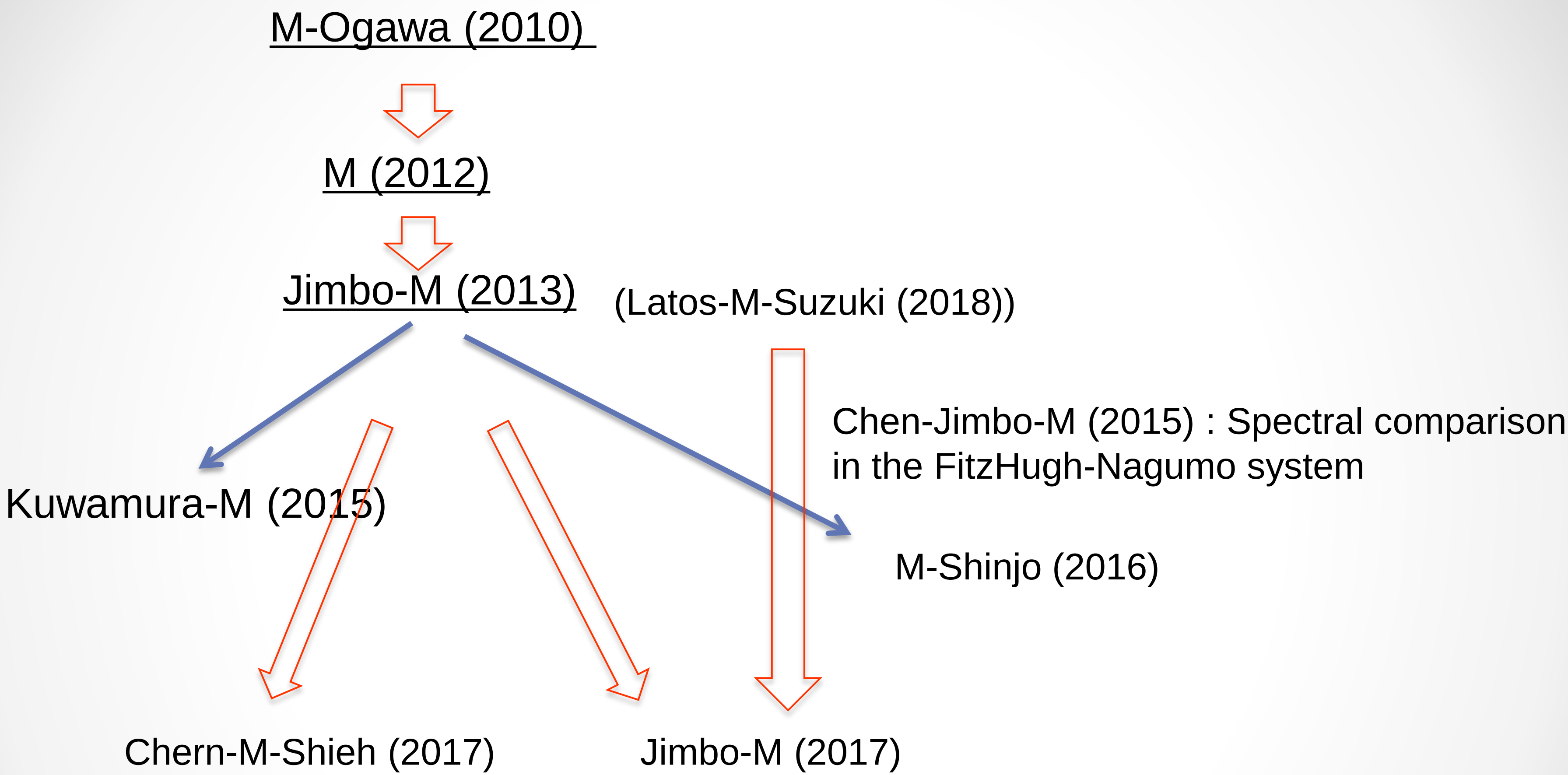
Y.Mori-Jikkine-etal (2011), SIAM J.Appl. Math.

Kuto-Tsujikawa (2013)

T.Mori-Kuto-Tsujikawa-Nagayama-Yotsutani (2015)

T.Mori-Kuto-Tsujikawa-Yotsutani (2016)

No results for the stability of the nonconstant solutions.


M-Ogawa (2010)
M (2012)
Jimbo-M (2013)
(Latos-M-Suzuki (2018))
Chen-Jimbo-M (2015) : Spectral comparison
in the FitzHugh-Nagumo system
Kuwamura-M (2015)
M-Shinjo (2016)
Chern-M-Shieh (2017)
Jimbo-M (2017)

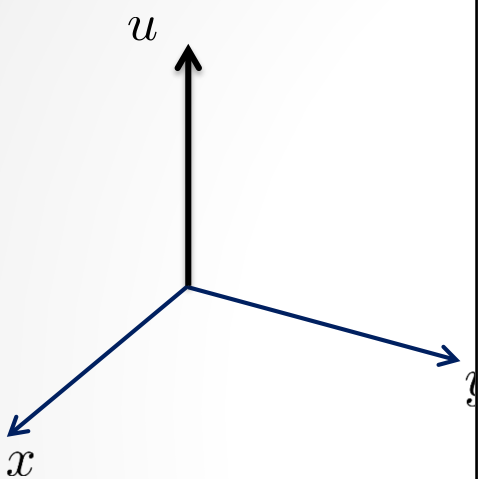
2. Mathematical motivation

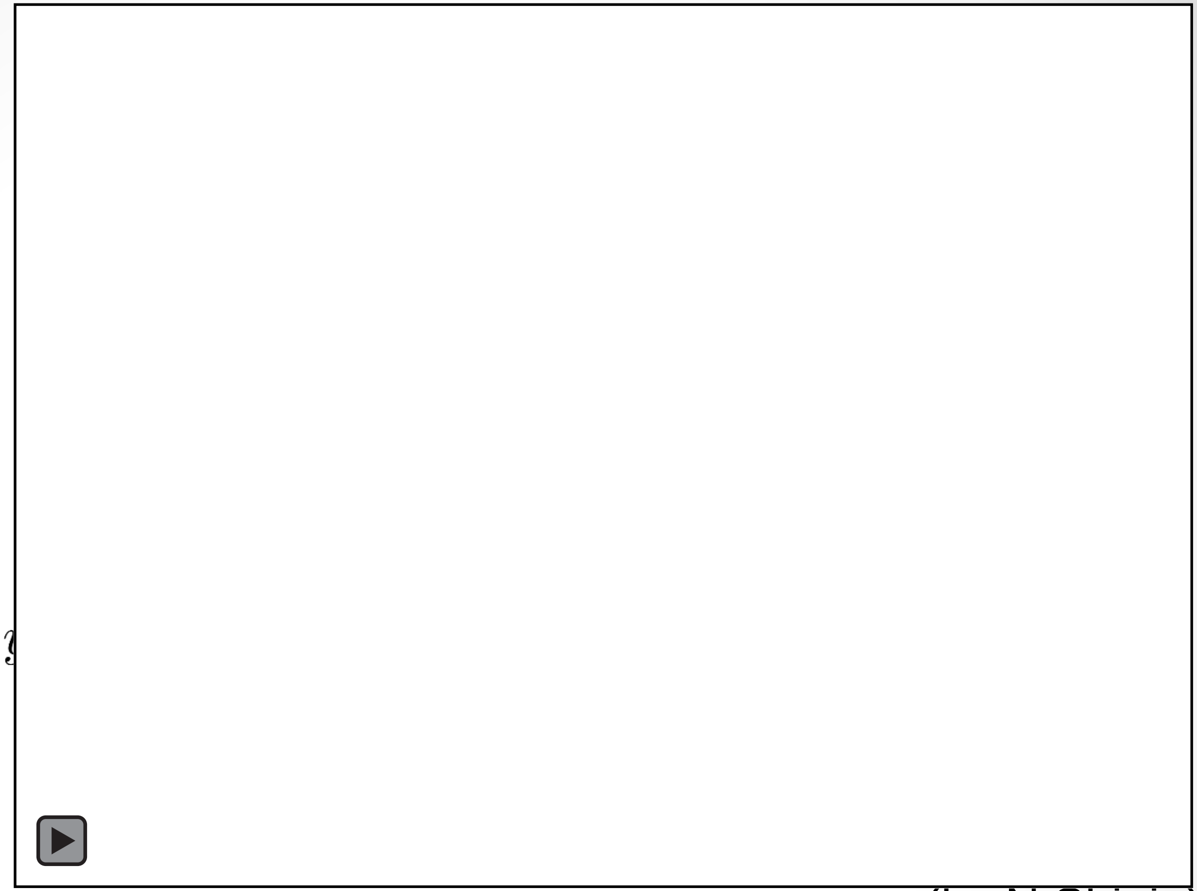
$$\gamma = 0, \quad g(u) = \frac{au}{u^2 + b}$$

Type I
$$\begin{cases} u_t = d_1 u_{xx} - \dfrac{au}{u^2+b} + v, \\[2ex] v_t = d_2 v_{xx} + \dfrac{au}{u^2+b} - v, \end{cases}$$

$$\gamma = 1, \quad g(w) = \frac{w}{(aw+1)^2}$$

Type II
$$\begin{cases} u_t = d_1 u_{xx} - \dfrac{u+v}{(a(u+v)+1)^2} + v, \\[2ex] v_t = d_2 v_{xx} + \dfrac{u+v}{(a(u+v)+1)^2} - v, \end{cases}$$





(by N.Shinjo)

$$\Omega = (0,3) \times (0,3)$$

$$u_0(x) = 1 + \cos(4\pi x)\cos(4\pi y),$$
$$v_0(x) = 1 + \cos(2\pi x)\cos(2\pi y)$$

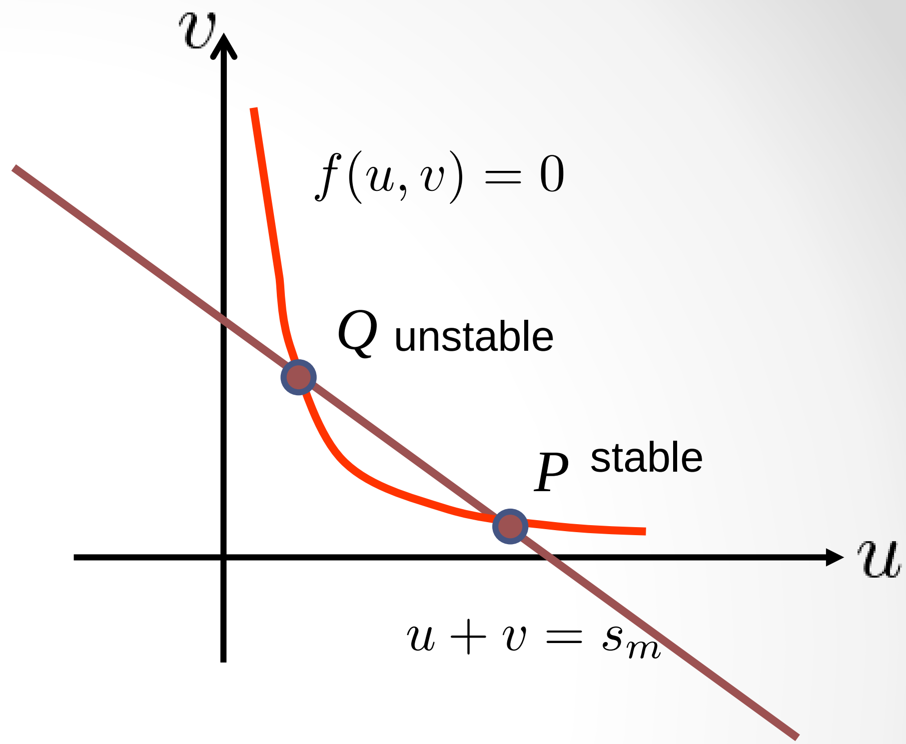
<u>Neumann boundary condition</u>

$$d_1 = 0.02, \quad d_2 = 1$$

$$\begin{cases} \dot{u} = f(u,v), \\ \dot{v} = -f(u,v). \end{cases}$$

$$\frac{d}{dt}(u+v) = 0$$

$$u + v = constant$$



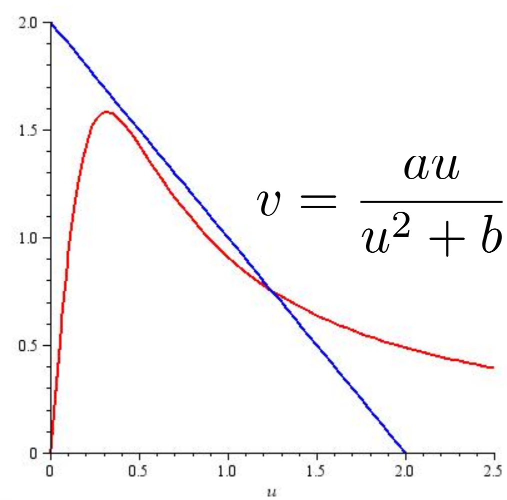
Type I

$$f = -\frac{au}{u^2+b} + v$$

$$a = 1, \quad b = 0.1$$

$$u + v = 2$$

There is a unique stable equilibrium.



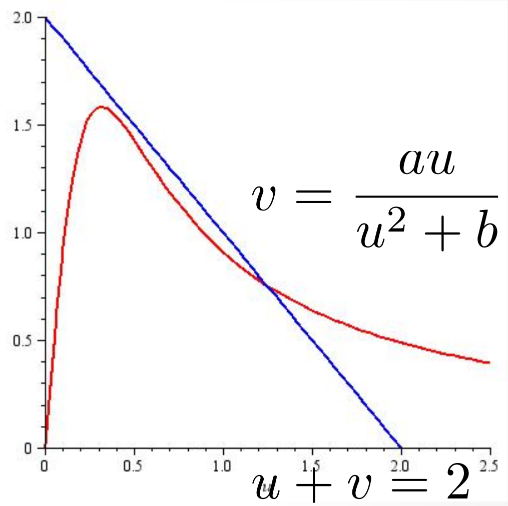
$u + v = 2$

For

$$f(u,v) = -\frac{au}{u^2+b} + v$$

Turing type instability takes place when

$$a = 1, \quad b \le \frac{1}{8}, \quad s_m \ge 2$$



The constant solution is no longer stable in the presence of diffusions.

Emergence of the wave pattern!

Type II:

$$\begin{cases} u_t = d_1 u_{xx} - g(u+v) + v, \\ v_t = d_2 v_{xx} + g(u+v) - v, \end{cases} \qquad g(w) = \frac{w}{(aw+1)^2}$$

$$s_m = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx + \frac{1}{|\Omega|}\int_\Omega v(x,t)dx$$

Unique constant steady state:

$$(u,v) = (s_m - g(s_m), g(s_m))$$

Assume $1 < as_m$ so that $g'(s_m) = \frac{1 - as_m}{(as_m+1)^3} < 0$ holds.

Then the constant solution is destabilized by the diffusions $0 < d_1/d_2 << 1$

(Q) How is the asymptotic pattern of $t \to \infty$ determined?
How is the profile of the pattern characterized for smaller diffusion d_1 ?

3. Spectral comparison in a generalized phase-field system

$$\begin{cases} \alpha u_t - \beta v_t = d\Delta u + f(u) + v, \\ u_t + \delta v_t = \Delta v \end{cases} \qquad (x \in \Omega)$$

$$\alpha > 0, \quad \beta \geq 0, \quad \delta \geq 0 \qquad d > 0$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \qquad (x \in \partial\Omega)$$

$$\Omega \subset \mathbf{R}^n \ : \text{bounded} \qquad \partial\Omega : \text{smooth boundary}$$

Conservation: $$\frac{d}{dt}\int_\Omega (u(x,t) + \delta v(x,t))dx = 0$$

Put

$$m := \langle u(\cdot,t)\rangle + \delta\langle v(\cdot,t)\rangle := \frac{1}{|\Omega|}\int_\Omega u(x,t)dx + \frac{\delta}{|\Omega|}\int_\Omega v(x,t)dx$$

Lyapunov function

$$\mathcal{E}(u,v) := \int_\Omega \left(\frac{d}{2}|\nabla u|^2 - F(u) + \frac{\beta}{2}|\nabla v|^2 + \frac{\delta}{2}v^2 \right) dx$$

$$\frac{d}{dt}\mathcal{E}(u(\cdot,t),v(\cdot,t)) = -\int_\Omega \left(\alpha u_t^2 + \beta\delta v_t^2 + |\nabla v|^2 \right) dx \le 0$$

The equation allows a gradient-like flow.

Stationary Problem:

$$\begin{cases} d\Delta u + f(u) + v = 0, \\ \Delta v = 0 \end{cases} \quad (x \in \Omega) \quad \text{+ Neumann B.C.}$$

$$\boxed{m = \langle u \rangle + \delta \langle v \rangle}$$

Case I: $\delta > 0$

$$v = \langle v \rangle = \frac{1}{\delta}(m - \langle u \rangle)$$

$$d\Delta u + f(u) + \frac{1}{\delta}(m - \langle u \rangle) = 0 \qquad (x \in \Omega) \qquad \text{+ Neumann B.C.}$$

Case II: $\delta = 0$

$$d\Delta u + f(u) - \lambda_f = 0 \qquad (x \in \Omega) \qquad \text{+ Neumann B.C.}$$

$$\langle u \rangle = m \qquad (\lambda_f = \langle f(u) \rangle = \langle v \rangle)$$

Variational structure

Case I: The Euler-Lagrange equation of

$$E_1(u) := \int_\Omega \frac{d}{2}|\nabla u|^2 - F(u)\,dx + \frac{|\Omega|}{2\delta}(m - \langle u\rangle)^2$$

$$F(u) = \int_0^u f(u)du$$

Case II: The Euler-Lagrange equation of

$$E_0(u) := \int_\Omega \frac{d}{2}|\nabla u|^2 - F(u)\,dx \qquad u \in H^1(\Omega), \quad \langle u\rangle = m$$

Given an equilibrium solution $u^*(x)$, we can define the Morse index.

(Q): How the stability and instability can be related to the system?

Spectral comparison principle

Bates-Fife (1990)

Cahn-Hilliard equation, Phase-field system

Ohnishi-Nishiura (1998)

Cahn-Hilliard equation

M (2012)

Phase-field system

Comparison between the original linearized eigenvalue problem and a simpler problem.

Both problems allow variational characterizations.

Linearized eigenvalue problem:

Case I Let $(u,v)=(u^*,v^*), \quad v^*=\frac{1}{\delta}(m-\langle u^*\rangle)$

be an equilibrium solution to the system.

$$\mathcal{A}_\delta \begin{pmatrix} \phi \\ \psi \end{pmatrix} := -\begin{pmatrix} d\Delta\phi + f'(u^*)\phi + \psi \\ \Delta\psi \end{pmatrix} = \lambda \begin{pmatrix} \alpha & -\beta \\ 1 & \delta \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Namely, $$-(d\Delta\phi + f'(u^*)\phi + \psi) = \lambda(\alpha\phi - \beta\psi)$$

$$-\Delta\psi = \lambda(\phi + \delta\psi)$$

$$(\phi,\psi) \in (H^2(\Omega))^2, \qquad \partial\phi/\partial\nu = \partial\psi/\partial\nu = 0, \quad \langle\phi\rangle + \delta\langle\psi\rangle = 0$$

Put

$$\tilde{\mathcal{A}}_\delta := \begin{pmatrix} \alpha & -\beta \\ 1 & \delta \end{pmatrix}^{-1} \mathcal{A}_\delta$$

On the other hand, corresponding to $E_1(u)$

$$\mathcal{L}_\delta\varphi := -(d\Delta\varphi + f'(u^*)\varphi) + \frac{1}{\delta}\langle\varphi\rangle = \mu\varphi$$

Case II

Let $(u,v) = (u^*, v^*), \quad v^* = \langle f(u^*) \rangle$

be an equilibrium solution to the system.

$$\mathcal{A}_0 \begin{pmatrix} \phi \\ \psi \end{pmatrix} := - \begin{pmatrix} d\Delta\phi + f'(u^*)\phi + \psi \\ \Delta\psi \end{pmatrix} = \lambda \begin{pmatrix} \alpha & -\beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

Namely,

$$-(d\Delta\phi + f'(u^*)\phi + \psi) = \lambda(\alpha\phi - \beta\psi)$$
$$-\Delta\psi = \lambda\phi$$

$$(\phi, \psi) \in (H^2(\Omega))^2, \qquad \partial\phi/\partial\nu = \partial\psi/\partial\nu = 0, \quad \langle \phi \rangle = 0$$

Put

$$\tilde{\mathcal{A}}_0 := \begin{pmatrix} \alpha & -\beta \\ 1 & 0 \end{pmatrix}^{-1} \mathcal{A}_0$$

On the other hand, corresponding to $E_0(u)$

$$\mathcal{L}_0\varphi := -(d\Delta\varphi + f'(u^*)\varphi) = \mu\varphi$$

$$\varphi \in H^2(\Omega), \quad \partial\varphi/\partial\nu = 0, \quad \langle \varphi \rangle = 0$$

(Q) How can we apply the spectral comparison argument?

Limiting behavior of eigenvalues of the nonlocal operator

Let $u_\delta^* \ (\delta > 0)$ be a solution to

$$d\Delta u + f(u) + \frac{1}{\delta}(m - \langle u \rangle) = 0 \qquad (x \in \Omega)$$

+ Neumann B.C.

$\delta > 0$

We assume $u_\delta^* \to u_0^* \ (\delta \to 0)$ where u_0^* is a solution to

$$d\Delta u + f(u) - \lambda = 0 \qquad (x \in \Omega)$$

$$\langle u \rangle = m$$

+ Neumann B.C.

$$\mathcal{L}_\delta \varphi := -(d\Delta\varphi + f'(u_\delta^*)\varphi) + \frac{1}{\delta}\langle \varphi \rangle = \mu\varphi$$

(Q) $\mu = \mu_k(\delta) \to \mu_k(0) \ \ (?) \qquad (\delta \to 0)$

$$\mathcal{L}_0 \varphi := -(d\Delta\varphi + f'(u_0^*)\varphi) = \mu\varphi \qquad \langle \varphi \rangle = 0$$

Case I

Prove the following:

1) If $\mathrm{Re}\lambda < \dfrac{1}{2\beta}$, then λ is real.

2) For $\lambda \leq 0$

$$\mathrm{Ker}(\tilde{\mathcal{A}}_1 - \lambda I) = \mathrm{Ker}(\tilde{\mathcal{A}}_1 - \lambda I)^m \quad (m \geq 2)$$

3) Compare the number of negative eigenvalues of $\tilde{\mathcal{A}}_\delta$ and $\mathcal{L}_\delta$

Theorem (Jimbo-M)

1) The spectrum of $\tilde{\mathcal{A}}_1$ consist of eigenvalues and its eigenvalues in $\{\lambda \in \mathbf{C} : \mathrm{Re}\lambda < 1/2\beta\}$ are real.

2) Let n_1 be the number of negative eigenvalues of $\tilde{\mathcal{A}}_1$, counting multiplicity, and let n_2 be that of $\mathcal{L}_1$. Then $n_1 = n_2$ holds.
In addition, if $0 \in \sigma_p(\tilde{\mathcal{A}}_1)$, or ($0 \in \sigma_p(\mathcal{L}_1)$) , then

$$\mathrm{dimKer}(\tilde{\mathcal{A}}_1) = \mathrm{dimKer}(\mathcal{L}_1)$$

Corollary

If the critical point $u^*(x)$ is a non-degenerate local minimizer of $E_1(u)$, then the solution $(u^*(x), v^*(x))$ of the system is stable.
If it is a (global) minimizer, then the corresponding solution is stable.

Case II Similar results hold in the context of the mass-conserved Reaction-diffusion system (Jimbo-M 2013).

Remark

A partial result, namely under some condition for the parameter values, is given in Latos-M-Suzuki.

Sketch for the proof of Theorem A.

$$\mathcal{A}_1\begin{pmatrix}\phi\\ \psi\end{pmatrix} := -\begin{pmatrix} d\Delta\phi + f'(u^*)\phi + \psi \\ \Delta\psi \end{pmatrix} = \mu\begin{pmatrix}\alpha & -\beta \\ 1 & \delta\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}$$

By $\psi = \langle\psi\rangle + \psi^Q, \qquad \psi^Q = \mu(-\Delta - \delta\mu)^{-1}\phi^Q \quad (\langle\phi\rangle + \delta\langle\psi\rangle = 0)$

$$\mathcal{L}_1(\phi) = \mu[(\alpha + \beta/\delta)\langle\phi\rangle + \{\alpha + (1 - \beta\mu)(-\Delta - \delta\mu)^{-1}\}\phi^Q]$$

$$\phi \in \{w \in H^2(\Omega) : \partial w/\partial\nu = 0 \ (x \in \partial\Omega)\})$$

Consider the eigenvalue problem with a parameter s :

$$\mathcal{L}_1(\phi) = \omega M(s)\phi$$

$$M(s)\phi := (\alpha + \beta/\delta)\langle\phi\rangle + \{\alpha + (1 + \beta s)(-\Delta + \delta s)^{-1}\}\phi^Q \quad (s \geq 0)$$

Let $\{\omega_k(s)\}$ be the set of eigenvalues,

$$\omega_1(s) \leq \omega_2(s) \leq \dots$$

We see

$$\mu_k \omega_k(0) > 0 \quad \text{if} \quad \omega_k(0) \neq 0 \quad (\text{or } (\mu_k \neq 0)) \qquad (\mu_k : \text{eigenvalues of } \tilde{\mathcal{L}}_1)$$

Find s^* such that

$$\mathcal{L}_1(\phi) = \omega(s^*) M(-\omega(s^*))\phi$$

Namely, $$s^* = -\omega(s^*)$$

Then $\omega(s^*)$ gives an eignvalue of $\tilde{\mathcal{A}}_1$.

Remark:

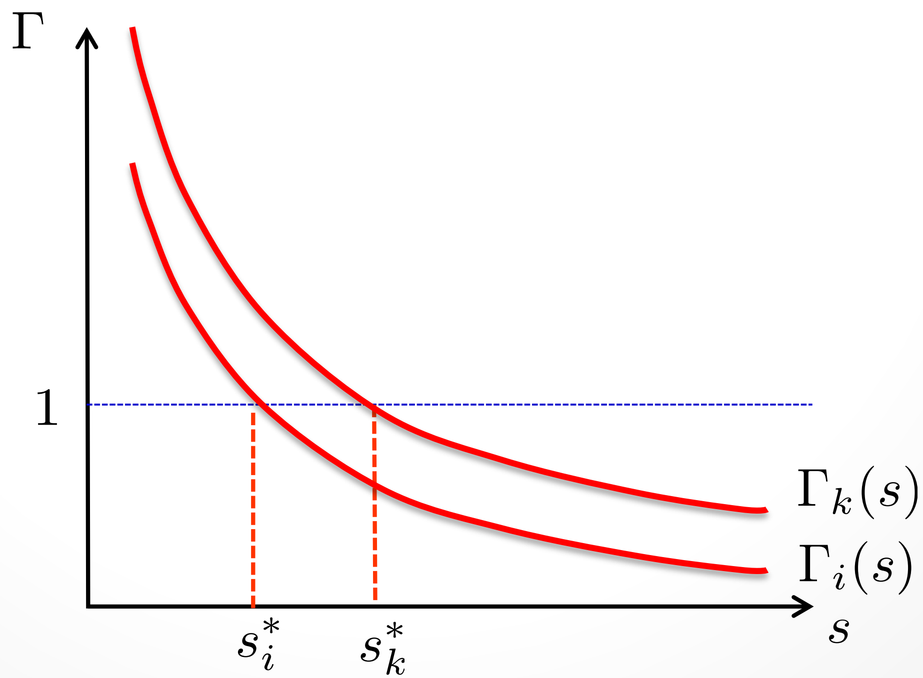
The monotonicity of $\omega_k(s)$ in s is ensured under some condition for the parameters (Latos-M-Suzuki)

Proposition

Each $\omega_k(s)$ is continuous in $s \geq 0$.
Moreover, if $\omega_k(0) < 0$, then $\omega_k(s)/s$ is strictly monotone increasing while if $\omega_k(0) > 0$, then $\omega_k(s)/s$ is strictly monotone decreasing.

Put $\Gamma_k(s) := -\omega_k(s)/s$ for $\omega_k(0) < 0$

Then there is a unique $s^* > 0$ such that $\Gamma_k(s^*) = 1$, and then $\omega_k(s^*)$ gives an eigenvalue of the problem.



Continuity of the eigenvalues in the parameter.

Nonlocal eigenvalue problem

$$\mathcal{L}\phi = \omega(M(s)\phi, \phi)_{L^2}$$

$M(s)$: self adjoint

$\exists \eta > 0$ s.t. $(M(s)\phi, \phi)_{L^2} \geq \eta \|\phi\|^2$

Rayleigh quotient:

$$R[\phi; s] := \frac{K[\phi]}{(M(s)\phi, \phi)_{L^2}}, \qquad K[\phi] := d\|\nabla\phi\|^2 + (a(\cdot)\phi, \phi)_{L^2}$$

$\mathcal{M}^n$: a set of all the n dimensional subspace in $L^2(\Omega)$

Variational characterization by the Min-Max principle:

$$\omega_n(s) = \inf_{X_n \in \mathcal{M}^n} \sup\{R[\phi; s] : \phi \in X_n,\ \phi \neq 0\}$$

Define

$$\rho_M(s,s_0) := \|M(s_0)^{-1/2}(M(s)-M(s_0))M(s_0)^{-1/2}\|_{op}$$

Case: $\omega_k(s_0) > 0$

$$(1-\rho_M(s,s_0))\omega_k(s) \le \omega_k(s_0) \le (1+\rho_M(s,s_0))\omega_k(s)$$

Hence,

$$\frac{\omega_k(s_0)}{1+\rho_M(s,s_0)} \le \omega_k(s) \le \frac{\omega_k(s_0)}{1-\rho_M(s,s_0)} \qquad (|s-s_0| << 1)$$

Case: $\omega_k(s_0) < 0$

$$\frac{\omega_k(s_0)}{1-\rho_M(s,s_0)} \le \omega_k(s) \le \frac{\omega_k(s_0)}{1+\rho_M(s,s_0)} \qquad (|s-s_0| << 1)$$

Remark: We don’t have to assume the simplicity of $\omega_k(s)$.

Monotonicity of $\omega_k(s)/s$ in the parameter.

$$(s_1 M(s_1)\phi,\phi)_{L^2} < (s_2 M(s_2)\phi,\phi)_{L^2} \quad (\phi \neq 0) \quad s_1 < s_2$$

Key idea to prove the moonotonicity of $\omega_k(s)/s$ in the parameter.

$$(s_1 M(s_1)\phi, \phi)_{L^2} < (s_2 M(s_2)\phi, \phi)_{L^2} \quad (\phi \neq 0) \quad s_1 < s_2$$

follows from

$$\frac{d}{ds}(sM(s)) = \frac{d}{ds}[s(1+\beta s)(-\Delta + \delta s)^{-1}]$$

$$= (1+2\beta s)(-\Delta + \delta s)^{-1} - \delta(s + \beta s^2)(-\Delta + \delta s)^{-2}$$

$$= [(1+2\beta s)(-\Delta) + \delta\beta s^2](-\Delta + \delta s)^{-2} > 0$$

Sketch of the prove the continuity

$$\|\phi\|_s^2 - \|\phi\|_{s_0}^2 := (M(s)\phi,\phi)_{L^2} - (M(s_0)\phi,\phi)_{L^2}$$

$$= ([M(s) - M(S_0)]M(S_0)^{-1/2}M(S_0)^{1/2}\phi, M(S_0)^{-1/2}M(S_0)^{1/2}\phi)_{L^2}$$

$$= ([M(S_0)^{-1/2}(M(s) - M(S_0))M(S_0)^{-1/2}]M(S_0)^{1/2}\phi, M(S_0)^{1/2}\phi)_{L^2}$$

thus

$$(1-\rho_M(s,s_0))\|\phi\|_{s_0}^2 \le \|\phi\|_s^2 \le (1+\rho_M(s,s_0))\|\phi\|_{s_0}^2$$

$$\begin{aligned}\omega_k(s_0) = & \inf_{V\in\mathcal{M}^k} \sup\left\{\frac{K[\phi]}{\|\phi\|_{s_0}^2} : \phi \in V, \phi \neq 0\right\} \\ \le & \inf_{V\in\mathcal{M}^k} \sup\left\{\frac{K[\phi]}{\|\phi\|_{s_0}^2} : \phi \in V, \phi \neq 0, K[\phi] \ge 0\right\} \\ \le & \inf_{V\in\mathcal{M}^k} \sup\left\{(1+\rho_M(s,s_0))\frac{K[\phi]}{\|\phi\|_s^2} : \phi \in V, \phi \neq 0, K[\phi] \ge 0\right\}\end{aligned}$$

from which

$$\omega_k(s_0) \le (1+\rho_M(s,s_0))\omega_k(s)$$

On the other hand, there is $V_\varepsilon \in \mathcal{M}^k$

$$0 < \sup\left\{\frac{K[\phi]}{\|\phi\|_{s_0}^2} : \phi \in V_\varepsilon, \phi \neq 0\right\} < \omega_k(s_0) + \varepsilon$$

Utilizing

$$\frac{K[\phi]}{\|\phi\|_{s_0}^2} > (1 - \rho_M(s, s_0))\frac{K[\phi]}{\|\phi\|_s^2} > 0$$

yields

$$(1 - \rho_M(s, s_0)) \sup\left\{\frac{K[\phi]}{\|\phi\|_s^2} : \phi \in V_\varepsilon, \phi \neq 0, K[\phi] \geq 0\right\} < \omega_k(s_0) + \varepsilon$$

This implies

$$(1 - \rho_M(s, s_0))\omega_k(s) < \omega_k(s_0) + \varepsilon$$

Consequently,

$$(1 - \rho_M(s, s_0))\omega_k(s) \leq \omega_k(s_0) \leq (1 + \rho_M(s, s_0))\omega_k(s)$$

4. Type II model

$$\begin{cases} u_t = d\Delta u - g(u+v) + v, \\ v_t = \Delta v + g(u+v) - v \end{cases}$$

+ Neumann B.C

$$s_m = \langle u(\cdot)\rangle + \langle v(\cdot)\rangle$$

Introduce new variables: $w = u + v$

$z = du + v$

$$\begin{cases} (1+d)w_t - z_t = d\Delta w - (1-d)g(w) - dw + z, \\ w_t = \Delta z \end{cases}$$

$$s_m = \langle w \rangle$$

Lyapunov function:

$$\mathcal{E}_1(w,z) := \int_\Omega \left(\frac{d}{2}|\nabla w|^2 + (1-d)G(w) + \frac{d}{2}w^2 + \frac{1}{2}|\nabla z|^2 \right) dx$$

$$G(w) := \int_0^w g(s)ds$$

$$\frac{d}{dt}\mathcal{E}_1(w,z) = -\int_\Omega (1+d)|w_t|^2 + |\nabla z|^2 \; dx \le 0$$

Stationary Problem:

$$\begin{cases} d\Delta w - (1-d)g(w) - dw + z = 0, \\ \\ \Delta z = 0 \end{cases}$$

From the second equation $z = \lambda \ (\text{constant})$

In sequel

$$d\Delta w - (1-d)g(w) - dw + \lambda = 0$$

$$\lambda = (1-d)\langle g(w)\rangle + d\langle w\rangle (= (1-d)\langle v\rangle + ds_m)$$

This is the Euler-Lagrange equation of

$$E_1(w) := \int_\Omega \left\{ \frac{d}{2}|\nabla w|^2 + (1-d)G(w) + \frac{d}{2}w^2 \right\} dx$$

in the admissible set $W := \{w \in H^1 : \langle w\rangle = s_m\}$

$$G(w) := \int_0^w g(s)ds = \log(w+1) + \frac{1}{w+1} - 1$$

Remark:

$$E(w) := \int_\Omega \frac{\varepsilon^2}{2}|\nabla w|^2 + F(w)\,dx \qquad F = \frac{1}{4}(w^2-1)^2$$

$$W := \{w \in H^1 : \int_\Omega w dx = m\}$$

$$\varepsilon^2 \Delta w + f(w) = \lambda, \qquad f = -F' = w - w^3$$

Stationary problem of the Cahn-Hilliard equation.

$$w_t = -\Delta(\varepsilon^2 \Delta w + w - w^3)$$

References (including the gradient flow):

Modica (1987) Gurtin-Matano (1988) Wei-Winter (1998)

Bates-Dancer-Shi (1999) ……………………..

Rubinstein-Sternberg (1992) Bronsard-Stoth (1997)

Chen-Hilhorst-Logak (2010) ……………………..

If the domain is cylindrical, that is, $\Omega = (0, L) \times \Sigma, \quad \Sigma \subset \mathbb{R}^{n-1}$ (bdd. domain) a stable solution $w^*(x)$ must be monotone in the axial direction.

(Gurtin-Matano (1988))

If $\Omega = (0, L)$, the minimizer is monotone, and non-monotone solution is unstable.

Remark: For the periodic boundary condition stable solution is constant or unimodal.

Theorem (Jimbo-M 2013)

The Morse index of any critical point $w^*(x)$ of $E_1(w)$ is equal to the number of negative eigenvalues of the linearized EVP for the equilibrium solution $(u^*(x), v^*(x))$. Moreover those has the equal multiplicity of zero eigenvalue.

Corollary

If the critical point is a non-degenerate local minimizer, or a (global) minimizer, then the solution of the system is stable.

Asymptotic profile $\quad \Omega = (0,1) \quad d = \varepsilon^2 \quad (0 < \varepsilon << 1)$

Define $\kappa = \kappa_\varepsilon$ by solving $\varepsilon = \dfrac{(\log \kappa)^{1/2}}{\kappa^2}$

We may assume that the minimizer $w_\varepsilon(x)$ is monotone decreasing.

Set $\quad m = \int_0^1 w \; dx = \langle w \rangle$

Theorem (Chern-M-Shieh)

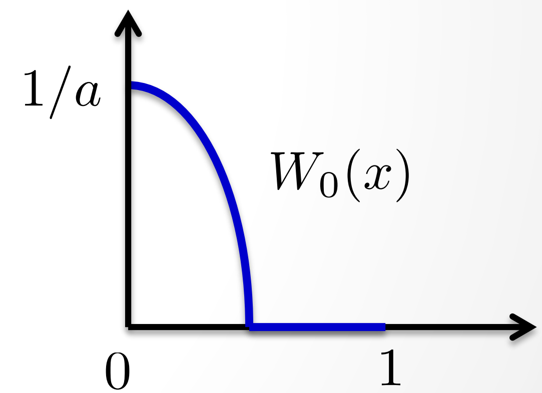
(i) $\{w_\varepsilon\} \to m\delta(x)$ in the sense:

$$\int_0^1 w_\varepsilon(x)dx = m \qquad \lim_{\varepsilon \to 0} \sup_{\eta \le x \le 1} w_\varepsilon(x) = 0 \quad (\forall \eta \in (0,1))$$

(ii) $\max_{0 \le x \le 1} w_\varepsilon(x) = w_\varepsilon(0) \le C_1 \kappa_\varepsilon$

(iii) $W_\varepsilon(x) := \dfrac{1}{\kappa_\varepsilon} w_\varepsilon\left(\dfrac{x}{\kappa_\varepsilon}\right) \to W_0(x) \quad (x \in \mathrm{R})$

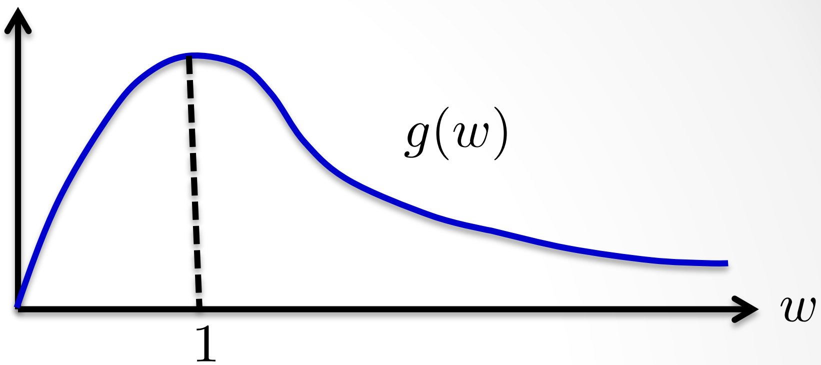
locally in $C^{0,\alpha}$ $(0 \le \alpha < 1)$



$$W_0(x) := \begin{cases} 1/a - ax^2/4 & (0 < x < \sqrt{2}/a) \\ 0 & (\sqrt{2}/a \le x \le 1) \end{cases} \qquad a := \left(\frac{2\sqrt{2}}{3m}\right)^{1/2}$$

Existence of unstable monotone solutions

$$g(w) = \frac{w}{(w+1)^2}$$



Under the condition

$$g'(s_m) = g'(m) < 0$$

$$\text{i.e., } m > 1$$

the uniform constant steady state $w = m$ is unstable in spatially inhomogeneous perturbation if ε is sufficiently small.

For $0 < m < 1$ the uniform constant steady state is stable for any ε.

Corollary

Assume $0 < m < 1$. Then there is (unstable) strictly monotone solution $\tilde{w}_\varepsilon(x)$ for sufficiently smal ε, and it satisfies

$$\liminf_{\varepsilon \to 0} E_\varepsilon(\tilde{w}_\varepsilon) > 0$$

Stability in the system

1) The minimizer $w_\varepsilon(x)$ gives an equilibrium

$$(u_\varepsilon^*(x), v_\varepsilon^*(x)) = \left(\frac{w_\varepsilon^*(x) - \varepsilon^2 s_m}{1-\varepsilon^2} - \langle g(w_\varepsilon^*) \rangle, \quad - \frac{\varepsilon^2 (w_\varepsilon^*(x) - s_m)}{1-\varepsilon^2} + \langle g(w_\varepsilon^*) \rangle \right)$$

to the system and it is stable in the system. This solution has a monotone profile.

2) Recall the constant solution $(\overline{u}, \overline{v}) = (m - g(m), g(m))$ to the system.

When $0 < m < 1, \quad i.e., \quad g'(m) > 0$, both $(u_\varepsilon^*(x), v_\varepsilon^*(x))$ and $(\overline{u}, \overline{v})$ are stable.

We can prove the instability for the equilibrium solution $(\tilde{u}_\varepsilon^*(x), \tilde{v}_\varepsilon^*(x))$ of the system given by $\tilde{w}_\varepsilon(x)$ which is unstable in the scalar problem.

End of Part I

In Part II I will give a sketch of the proof of Theorem (Chern-M-Shieh).