ナヴィエーストークス方程式に関連した爆発問題

岡本　久

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- Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Editors: Giga & Novotny to appear in Springer in 2018. Includes: O., Models and special solutions of the Navier-Stokes equations
- Bae, Chae & O., Nonlinear Analysis (2017)
- H. O., T. Sakajo, and M. Wunsch, On a generalization of the Constantin-Lax-Majda equation, *Nonlinearity* (2008)
- H.O., Well-Posedness of the Generalized Proudman-Johnson Equation Without Viscosity, *J. Math. Fluid Mech.* 0nline (2007)
- K. Ohkitani and H.O., *J. Phys. Soc. Japan*, **74** (2005), 2737--2742.
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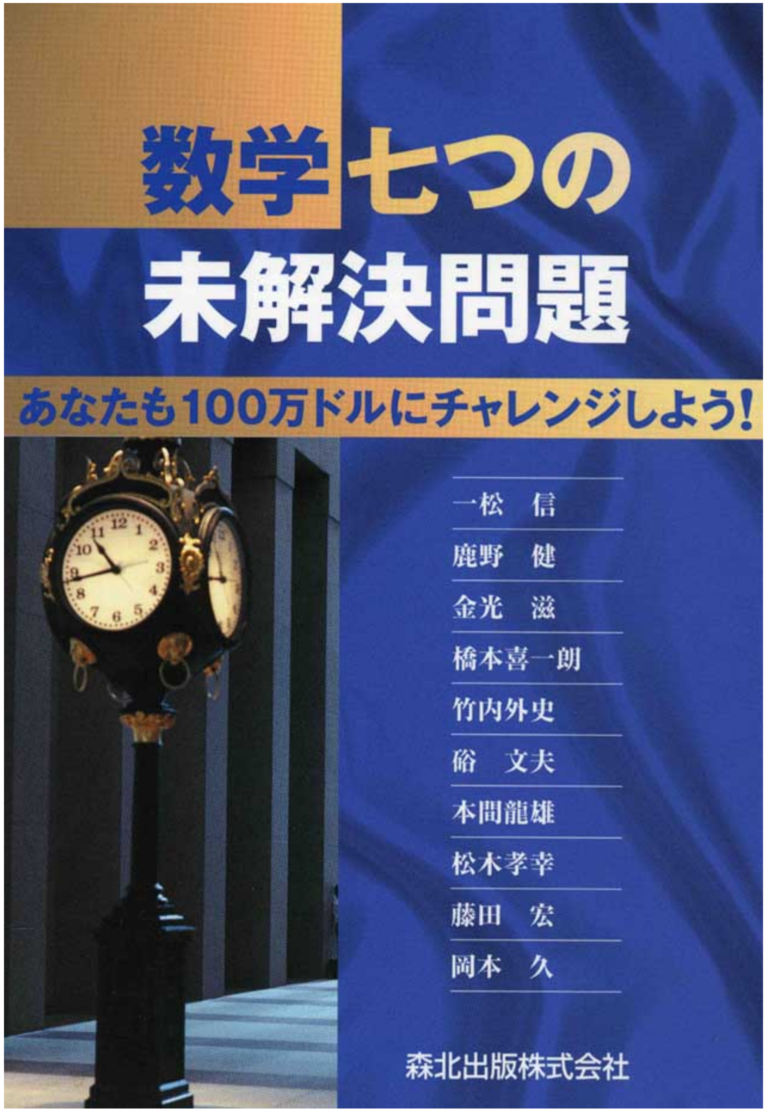
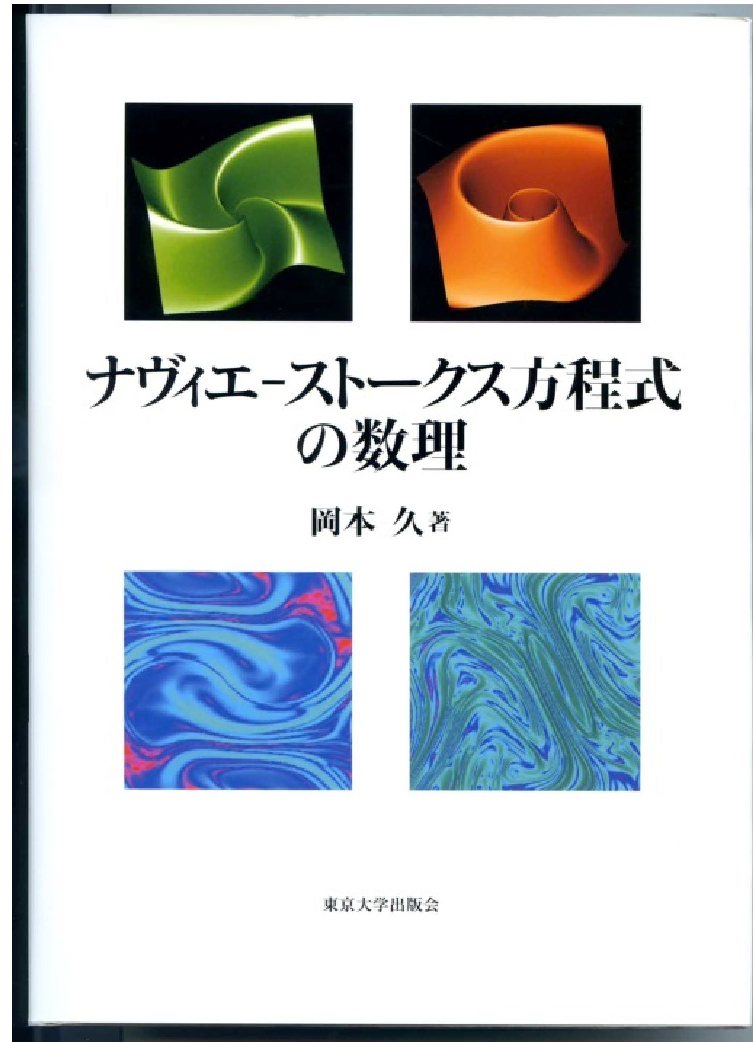
The Navier-Stokes equations

- Equations of motion of incompressible viscous fluid. 1827, 1845
- Many unsolved problems.



ナヴィエ・ストークスの解の正則性はなぜ面白いのか？

- 乱流の理論的理解
  - J. Lerayのシナリオ
  - L.D. Landauのシナリオ
  - Ruelle-Takenの仮説



The Navier-Stokes eqs.

- Incompressible viscous fluid
- **u** : velocity, p : pressure

➢ $\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p$

$\operatorname{div} \mathbf{u} = 0$

- ν : viscosity. $\nu = 0 \Rightarrow$ The Euler eqs.
- $\boldsymbol{\omega}_t + (\mathbf{u} \cdot \nabla)\, \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{u} = \nu \Delta \boldsymbol{\omega}$

convection **stretching** **viscousity**

$\boldsymbol{\omega} = \mathrm{curl}(\mathbf{u})$ --- vorticity

$$u(x) = -\frac{1}{4\pi} \int_{R^3} \frac{(x-y) \times \omega(y)}{|x-y|^3}\, dy$$

Heated arguments Kerr, Hou, ...

It is often said that

$$\boldsymbol{\omega}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \nabla)\mathbf{u} = 0$$

- The stretching term amplifies the vorticity;
- A nonlinear convection term may be a cause of singularities of shock wave type but it never magnifies the function;
- As far as the indefinite amplification of the vorticity, the convection term plays no positive role: it just sits and watches.

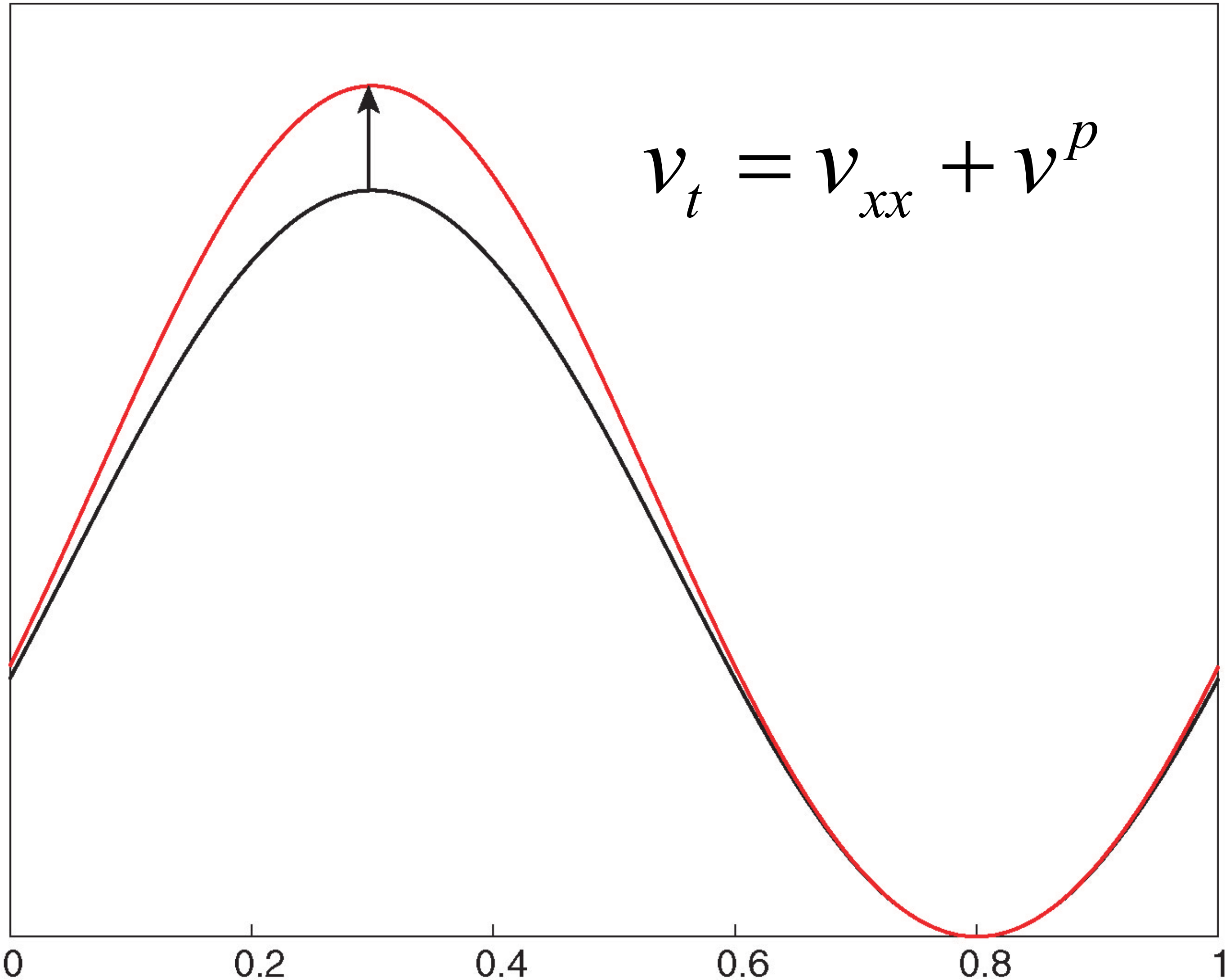
My goal today

- To show that the convection term plays a definite role in blow-up of solutions. The convection term may prevent vorticity from blowing-up.
- If a sol. becomes very large, stretching becomes ineffective: Depletion of nonlinearity.
- A case study for this claim

Deplete blow-up by convection

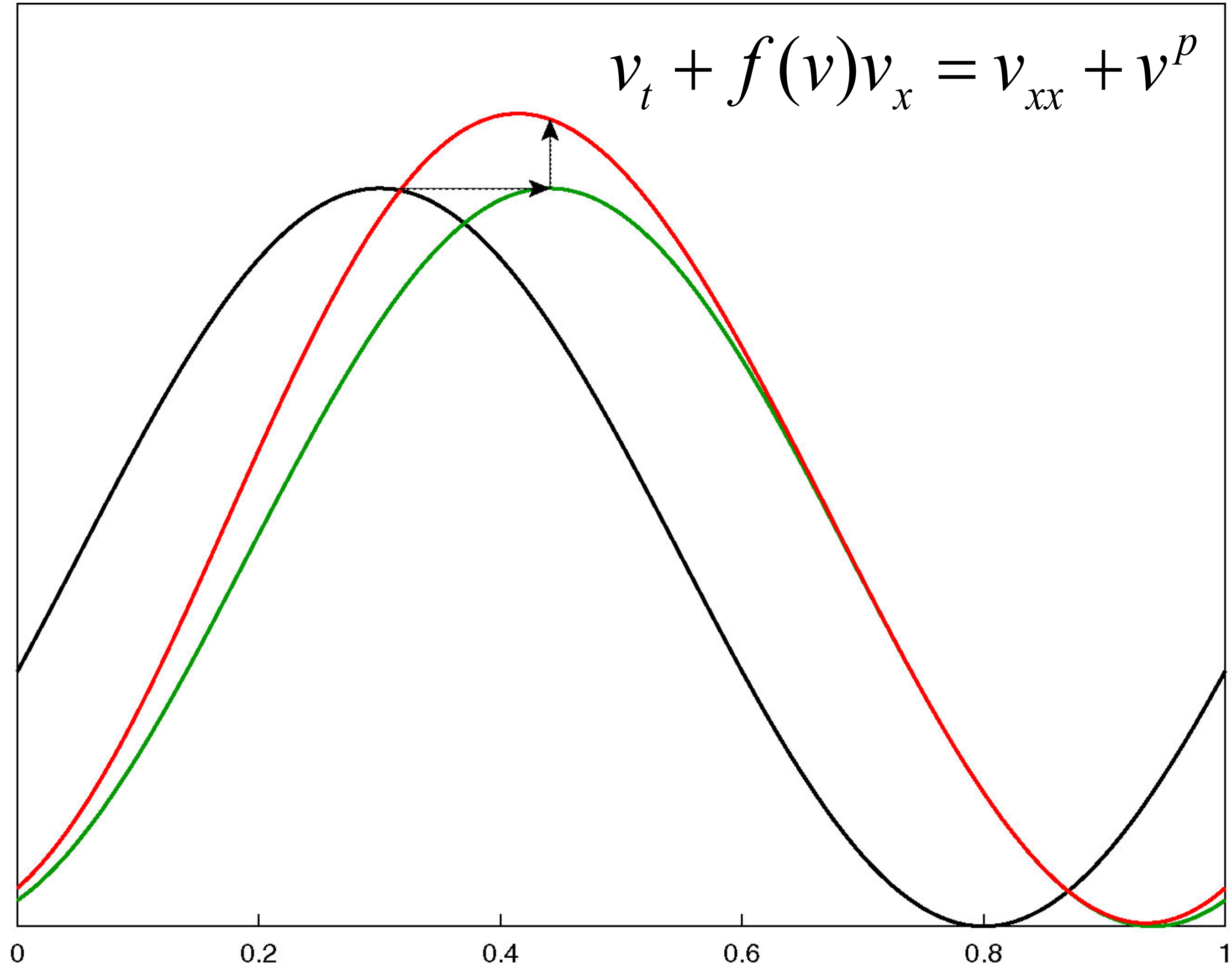
Naive explanation

- Without convection, the higher becomes higher still.

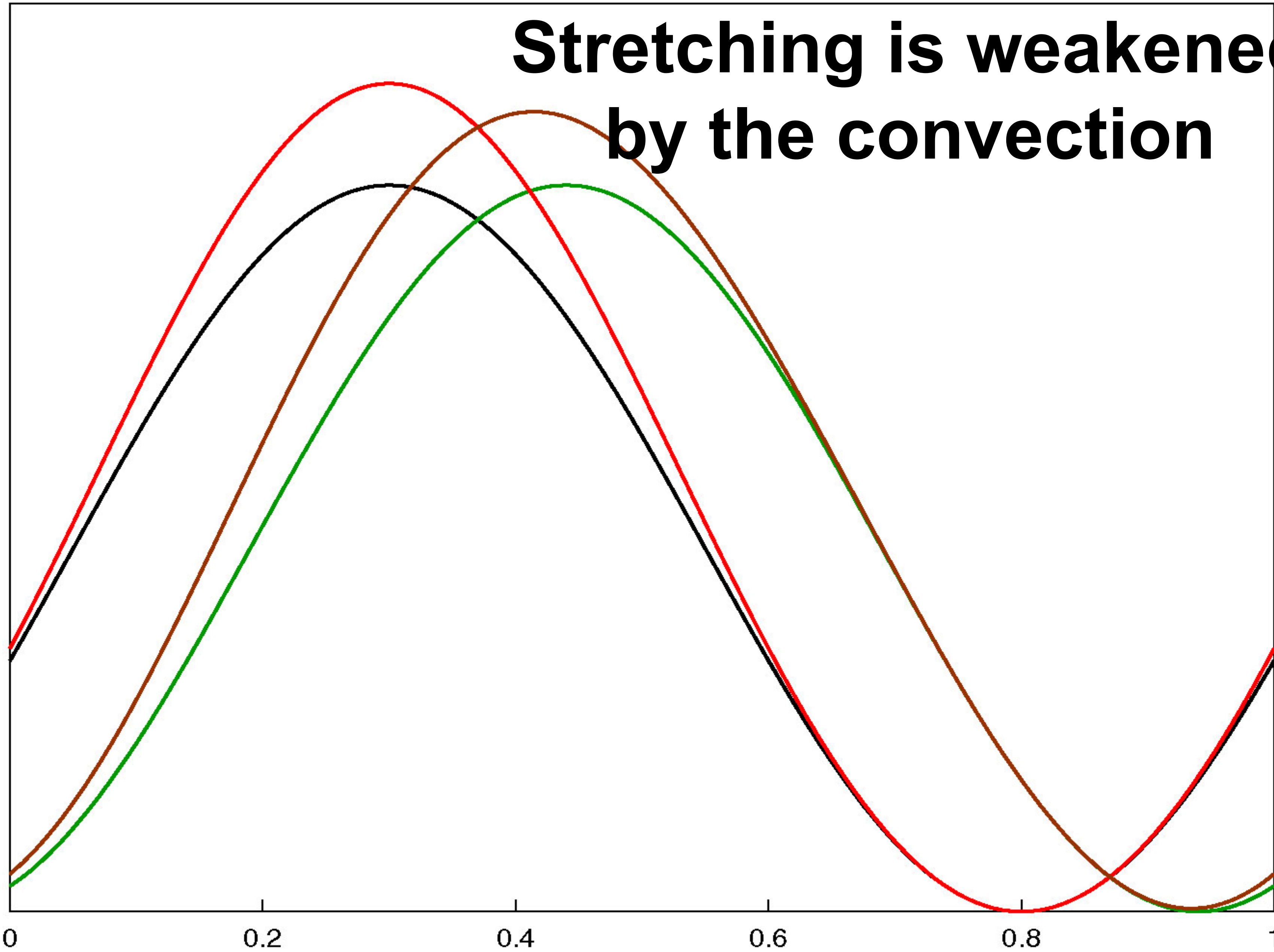


$$v_t = v_{xx} + v^p$$

- With convection, the highest does not necessarily become the highest.

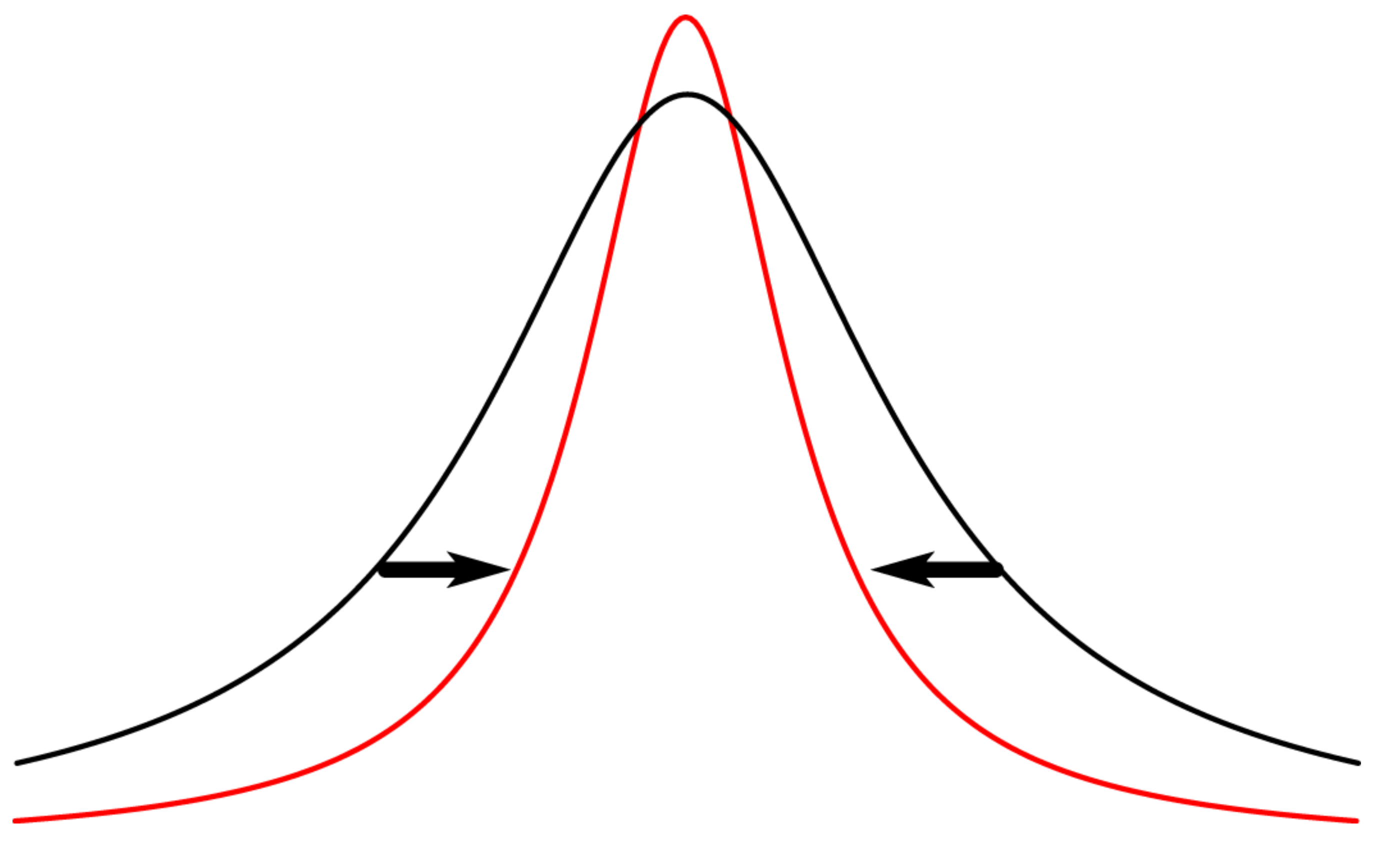


Stretching is weakened by the convection



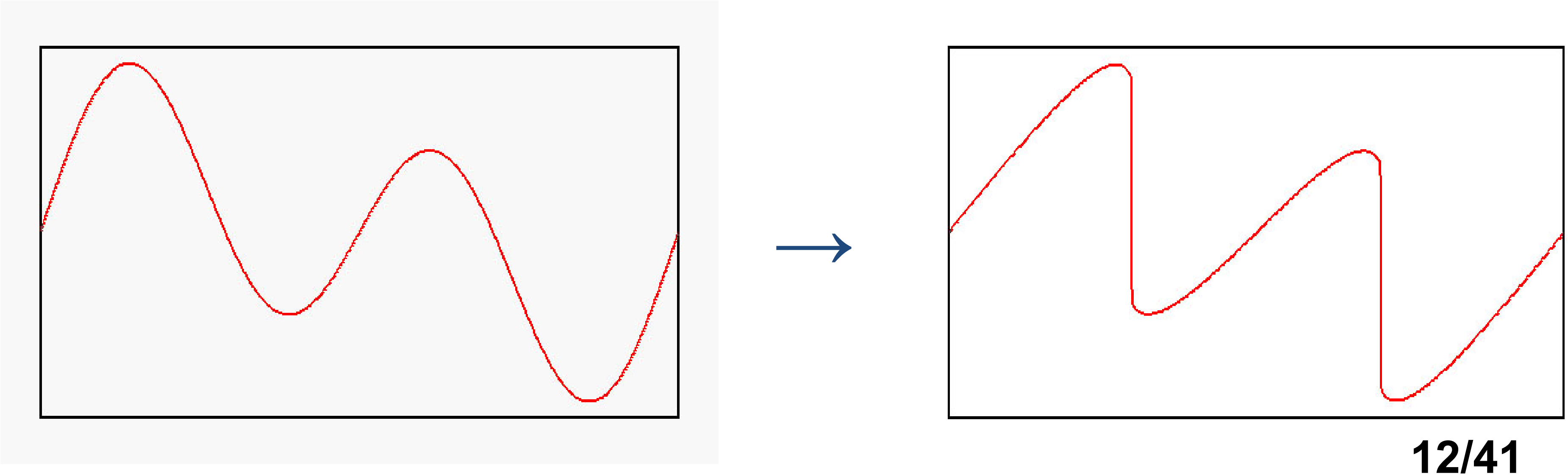
(unphysical & nonlinear) convection term may help blow-up

The direction of convection is important.



Introductory example or warming-up

- The Burgers eq. $u_t + uu_x = u_{xx}$
- Global existence; (not global if u_{xx} is missing)



Burgers eqn. continued

- Burgers eq. $u_t + uu_x = u_{xx}$
- $v = u_x$ (or ; $u = \int v dx$)

➢ $v_t + uv_x + v^2 = v_{xx}$ (no blow-up)

convection stretching viscosity

➢ $v_t + v^2 = v_{xx}$ (blow-up)

➢ $v_t - uv_x + v^2 = v_{xx}$ (blow-up)

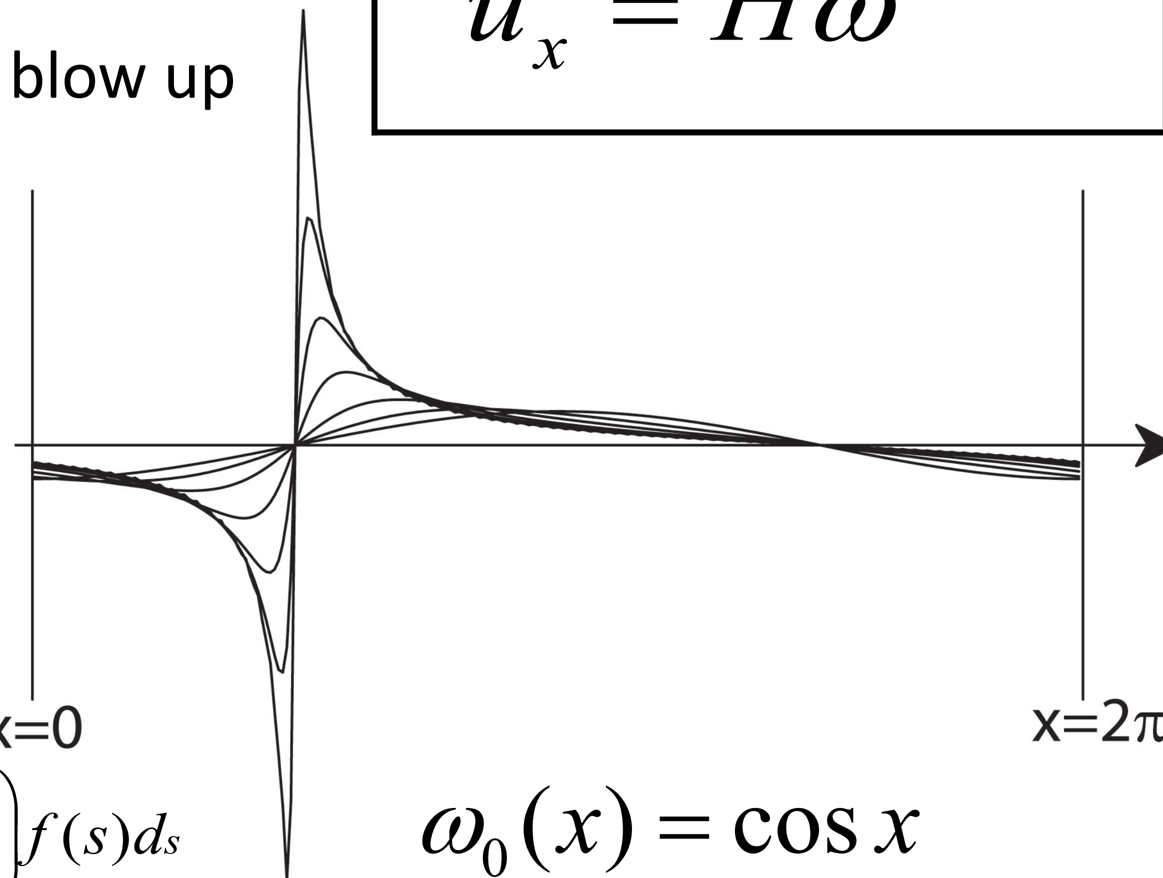
A thesis

- Viscosity alone is not enough to suppress the blow-up.
- But blow-up can be prevented by *viscosity* and/or an appropriate nonlinear *convection*.

Example①
Constantin-Lax-Majda ('85)

$$\omega_t - \omega u_x = 0$$
$$u_x = H\omega$$

Almost all solutions blow up in finite time.



$$Hf(\sigma) = \frac{1}{2\pi}\int_0^{2\pi} \cot\left(\frac{\sigma - s}{2}\right) f(s)\,ds$$

$$\omega_0(x) = \cos x$$

De Gregorio's equation

- S. De Gregorio, J. Stat. Phys. '90

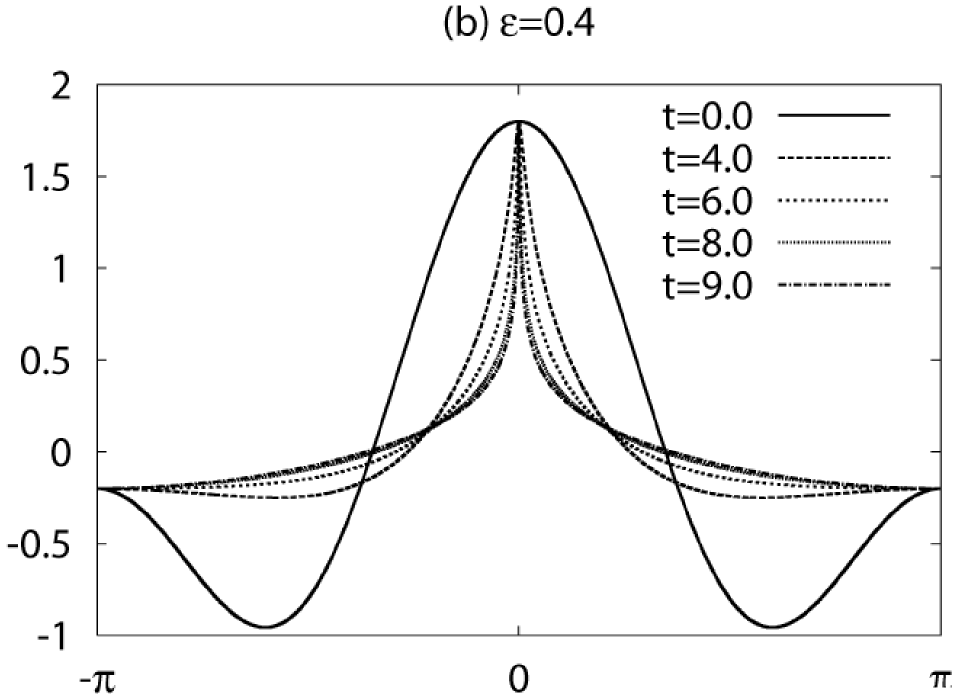
$$\omega_t + u\omega_x - \omega u_x = 0$$

$$u_x = H\omega$$

$$\omega_t + (\mathbf{u} \bullet \nabla)\omega - (\omega \bullet \nabla)\mathbf{u} = 0$$

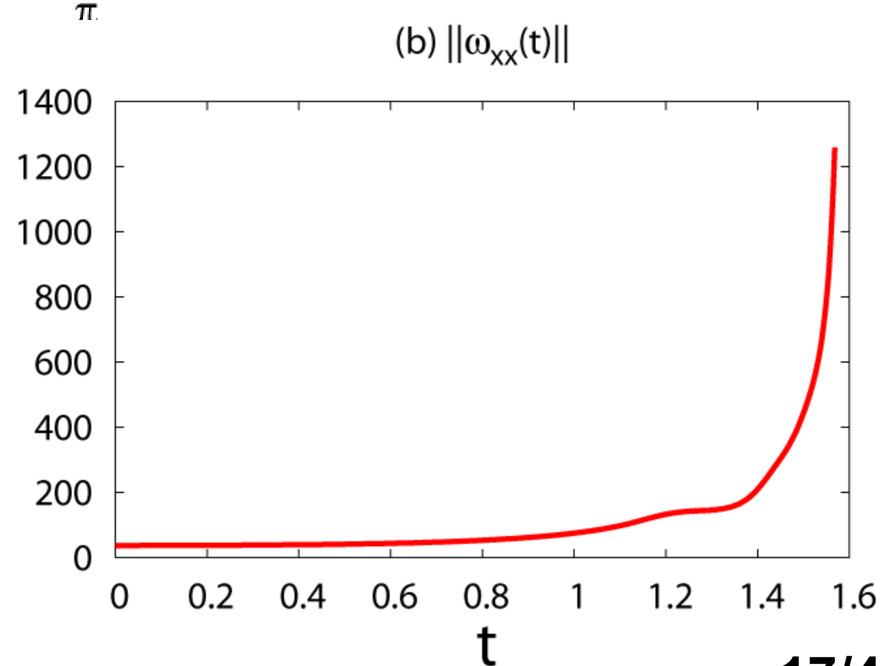
- Constantin-Lax-Majda eq. + convection term

- $\omega = \cos x$ is a steady-state.

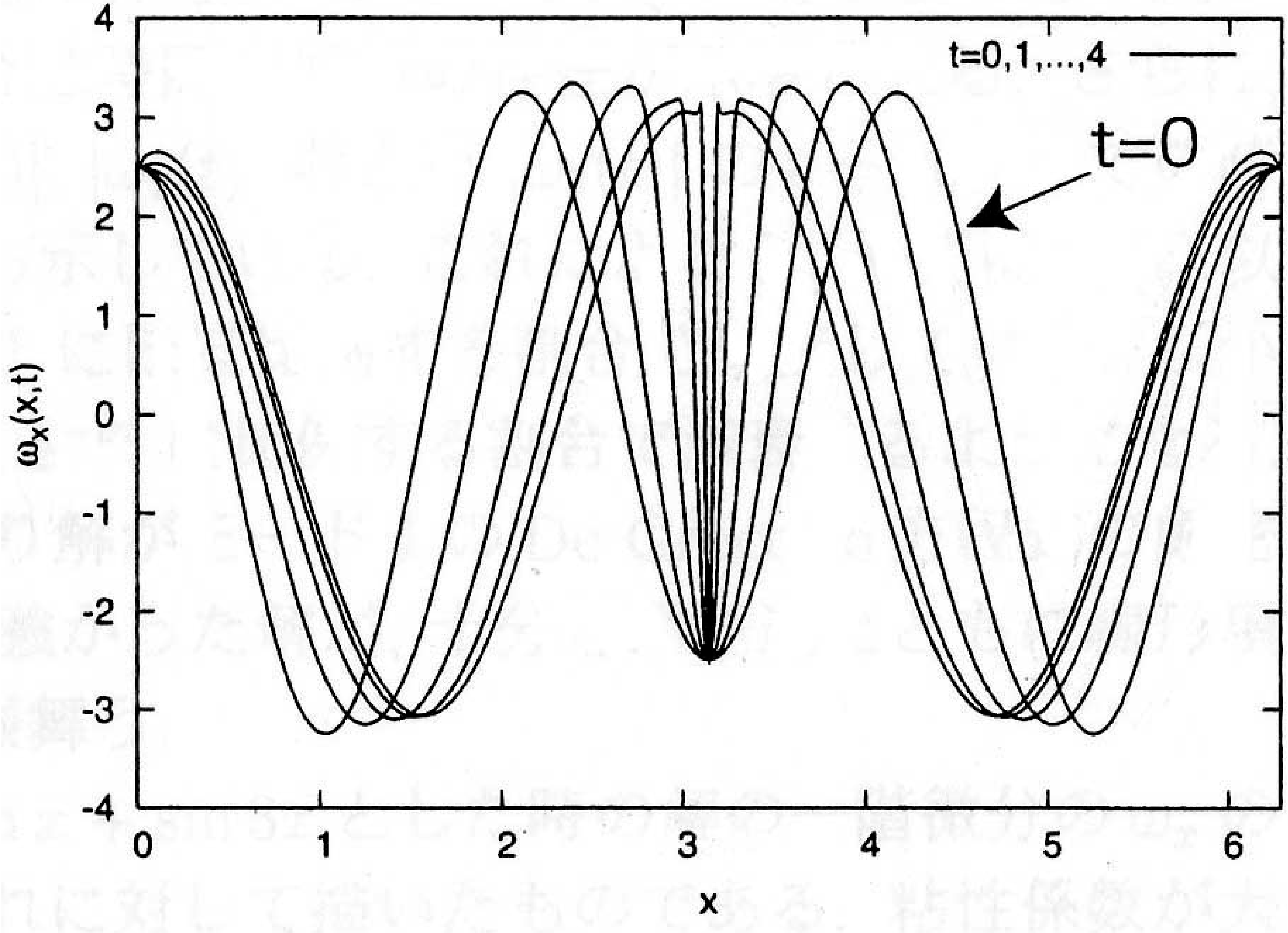


ω_x is bounded.

ω_{xx} grows rapidly.



Another exmaple



Local existence for De Gregorio

Theorem

If $\omega_0(x) \in H^1(S^1)$, then $\exists\ T_0 > 0$

such that a solution $\omega \in C([0,T_0];H^1(S^1))$.

Further,

$$\max_{0\le t\le T_0}\|\omega(t)\|_{H^1} \le C = C(T_{0,}\|\omega_0\|_{H^1})$$

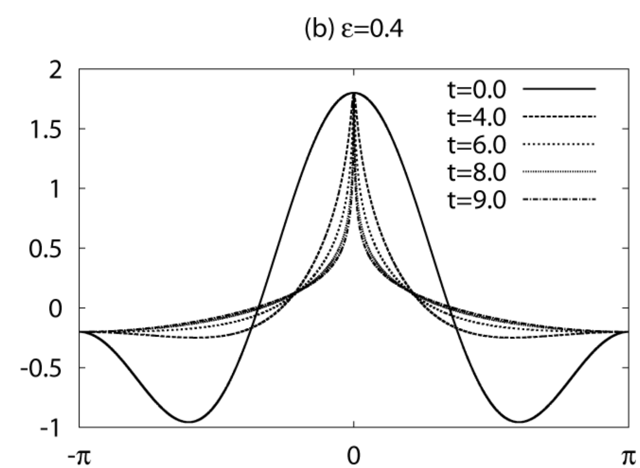
- Proof. Application of T. Kato's theory of nonlinear evolution equations

A sufficient condition for global existence

$$\int_0^T \left\| H\omega(t) \right\|_{L^\infty} dt < \infty \Rightarrow \text{solution exists in } [0, \mathrm{T} + \delta]$$

- An analogue of a theorem by Beale, Kato, & Majda '84.

Since $\left\| H\omega \right\|_\infty \leq c \left\| \omega_x \right\|_{L^2}$, boundedness $\Longrightarrow$ global existence.



A generalization

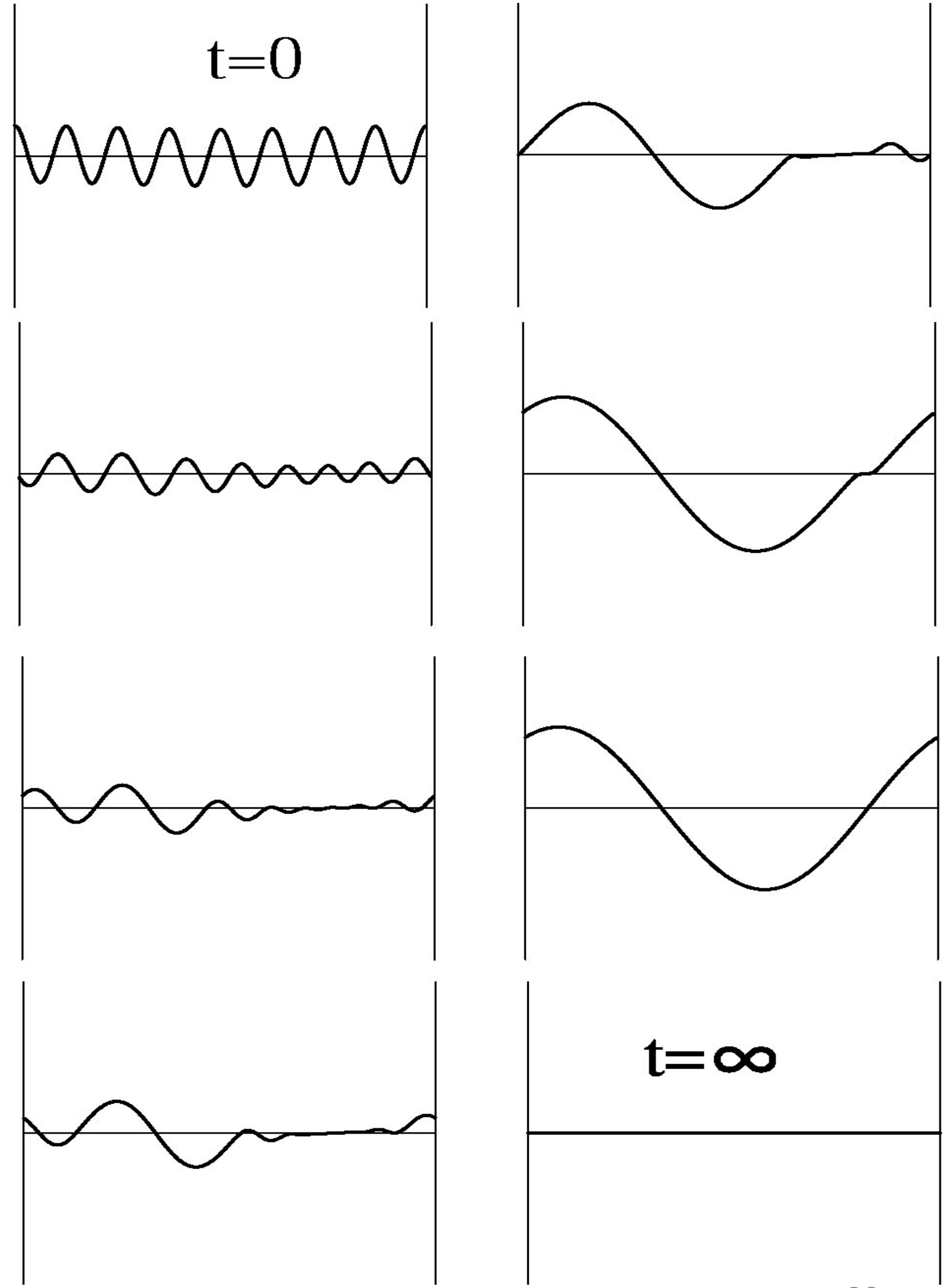
$$\omega_t + au\omega_x - \omega u_x = 0$$

$$u_x = H\omega$$

- a is an artificial parameter.

Theorem(?)
Blow-up if $-1 \leq a < 1$.
Global existence otherwise.

If viscosity is present,



Example ②

- 2D Euler ; incompressible in viscid

$$\omega_t + \mathbf{u} \cdot \nabla \omega = 0$$

$$\omega = \mathrm{curl}\ \mathbf{u} = v_x - u_y \ ; \ \mathbf{u} = (u,v)$$

$$\chi = \nabla \omega$$

$$\chi_t + (\mathbf{u} \cdot \nabla)\chi - (\chi \cdot \nabla)\mathbf{u} = \mathbf{0}$$

convection stretching

$$\chi = -\Delta \mathbf{u}$$

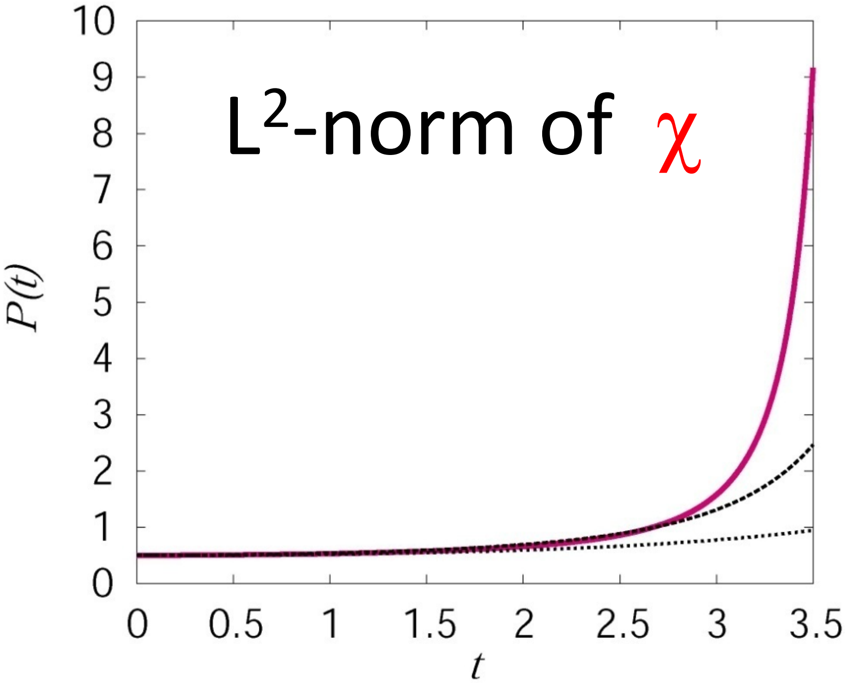
The convection term is now deleted.

- $\chi_t - (\chi \cdot \nabla)\mathbf{u} = \mathbf{0}$ in R^2

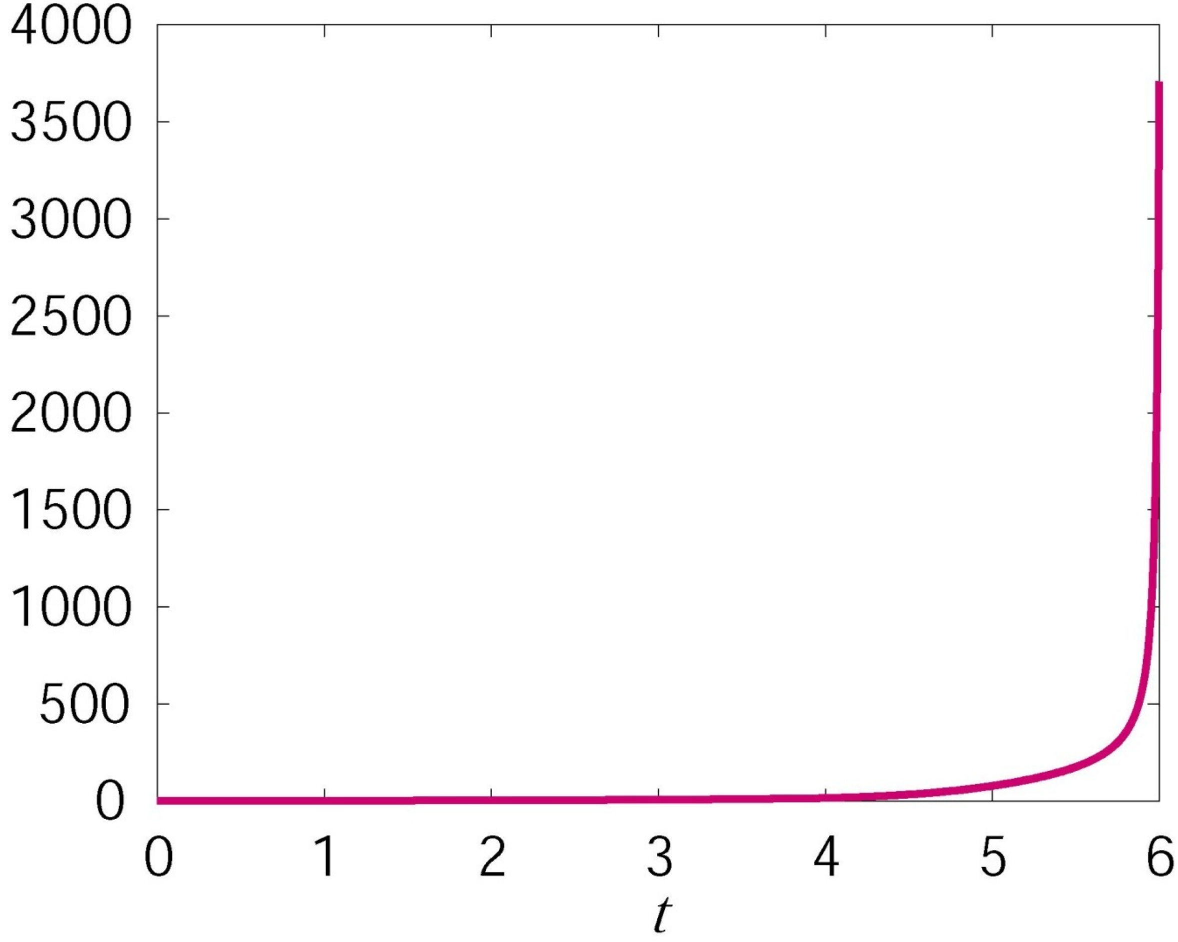
$\mathbf{u} = (-\Delta)^{-1}\chi$

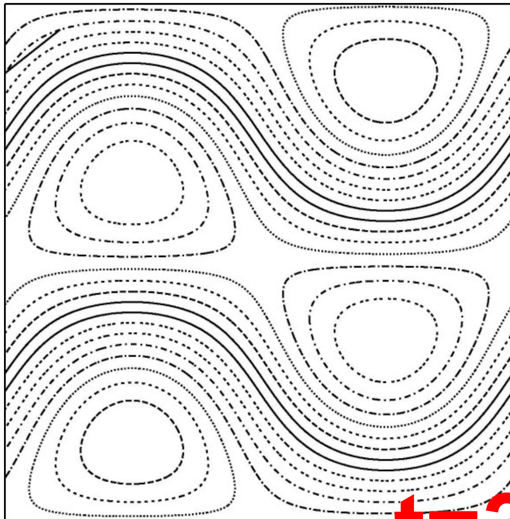
or

$\mathbf{u} = P(-\Delta)^{-1}\chi$



$$\int_0^t \|\chi(s)\|_\infty dx$$

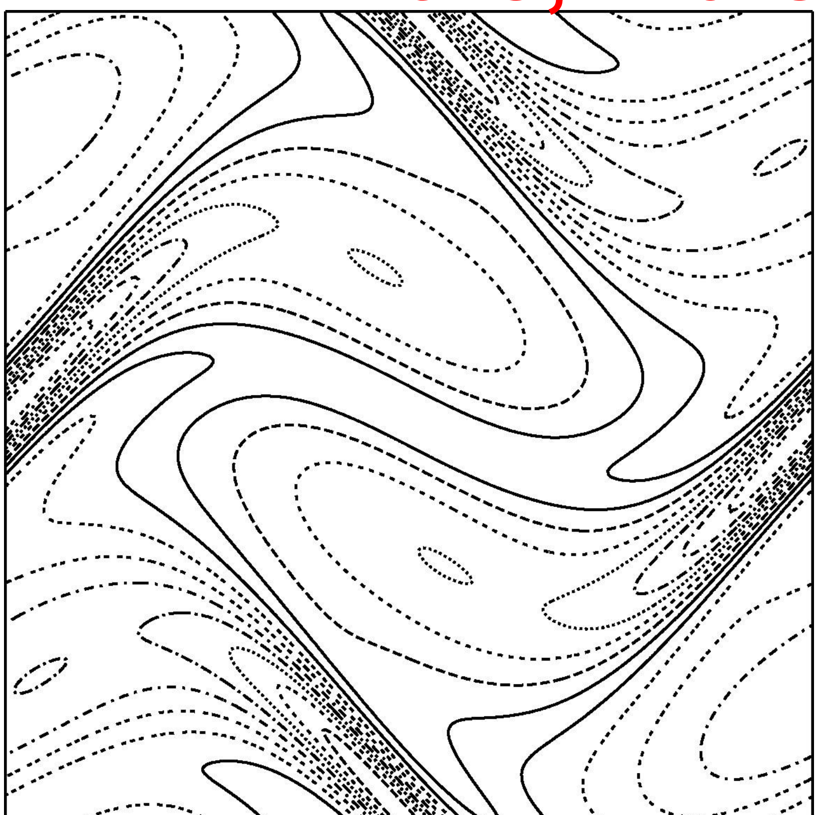




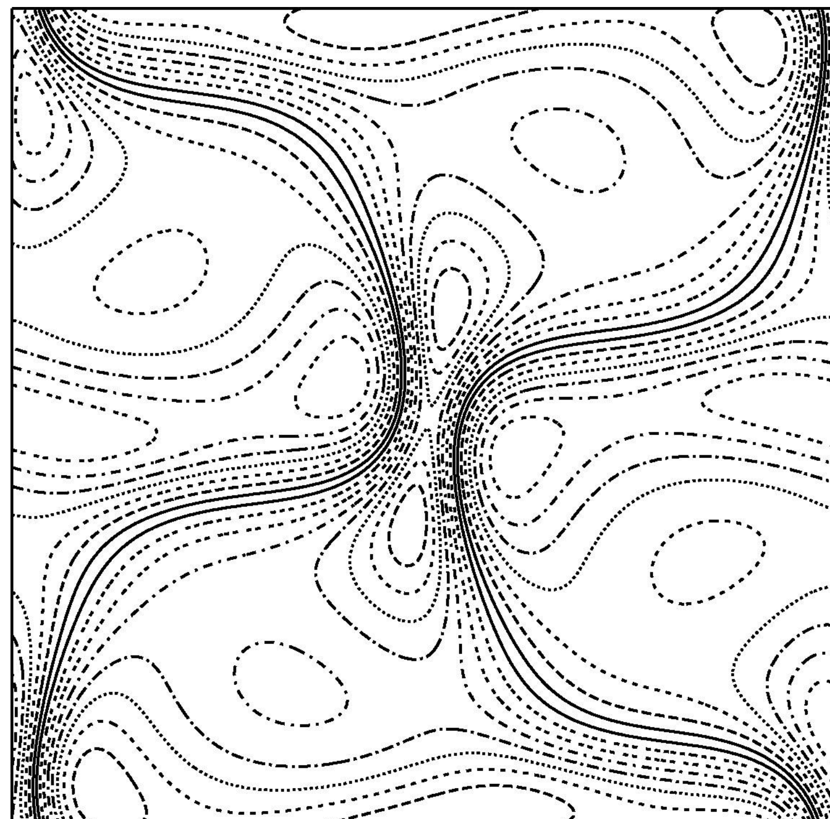
t=0

$$(-\Delta)^{1/2}\omega \sim |\chi|$$

t=3, Euler

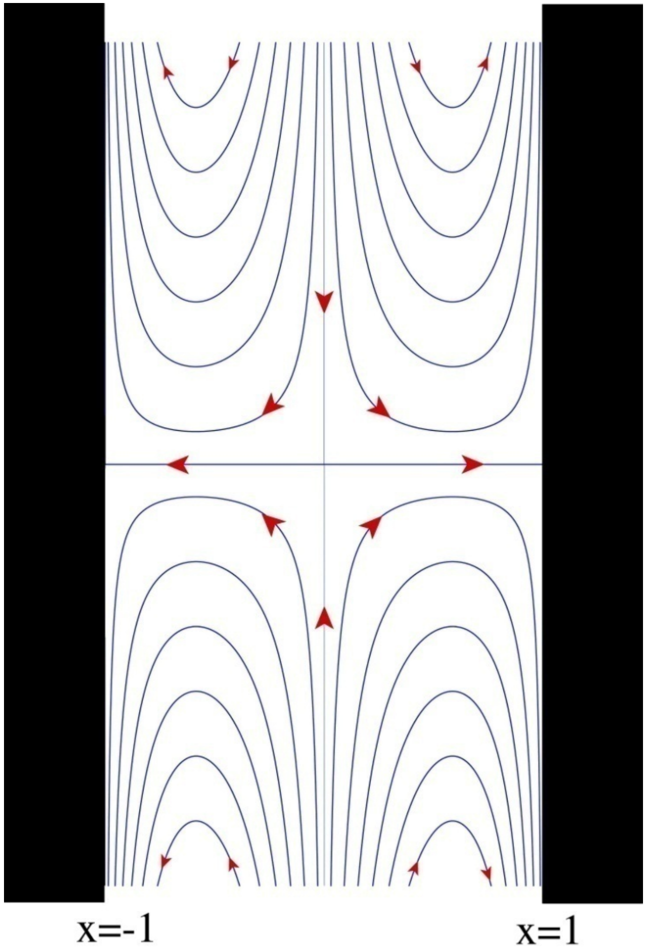


t=3, model



Example ③
The Proudman-Johnson equation

- Derived from 2D Navier-Stokes
- $f_{txx} + ff_{xxx} - f_x f_{xx} = \nu f_{xxxx}$
  $(0 < t, \quad -1 < x < 1)$
- $f(t, \pm 1) = 0, \quad f_x(t, \pm 1) = 0$
- $f_{xx}(t,x) = \phi(x)$
- $\mathbf{u} = (f(t,x), -yf_x(t,x))$
  (unbounded solution of NS)



Global existence or finite time blow-up?

- $f_{txx} + ff_{xxx} - f_x f_{xx} = \nu f_{xxxx}$

$\omega = f_{xx}$, $\qquad \omega_t + f\omega_x - f_x\omega = \nu\omega_{xx}$

- Had been difficult to judge

Global existence was proved by Xinfu Chen

THEOREM. Assume that $\nu > 0$.

For any initial data in $L^2(-1,1)$, a solution exists uniquely for all t and tends to zero as $t \to \infty$

Xinfu Chen and O., Proc. Japan Acad., 2000.

Effect of convection term

- $$f_{txx} + ff_{xxx} - f_x f_{xx} = \nu f_{xxxx}$$

$$\omega = f_{xx}, \qquad \omega_t + f\omega_x - f_x\omega = \nu\omega_{xx}$$

- $$f_{txx} - f_x f_{xx} = \nu f_{xxxx}$$

- $$f_{tx} - ½ f_x^{\,2} = \nu f_{xxx} + \beta \, ; \qquad u \equiv ½ f_x$$

- $$\boxed{u_t = \nu u_{xx} + u^2 + \beta}$$

$$u_t = u_{xx} + u^2 + \beta$$

$$(0 < t, -1 < x < 1)$$

$$\int_{-1}^{1} u(t,x)dx = 0, \quad u(t,\pm 1) = 0$$

$$u_t = u_{xx} + Pu^2, \quad P : L^2 \to L^2 / \mathbf{R}$$

blow-up occurs.

A proper convection term prevents solutions from blowing-up.

Budd, Dold & Stuart ('93), Zhu &O. ('00)

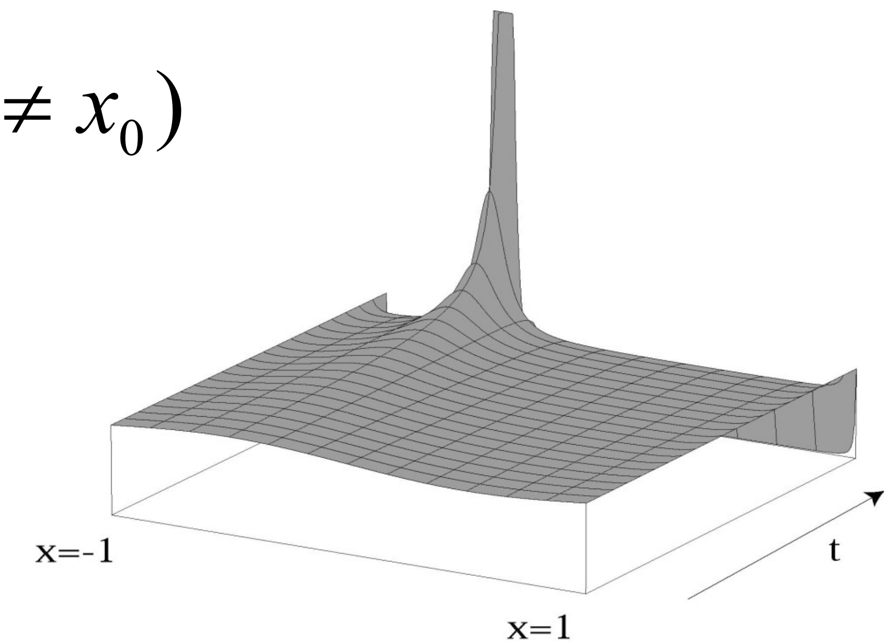
$$u_t = \nu u_{xx} + u^2 - \int_0^1 u(t,x)^2 \, dx.$$

- ∃ x_0

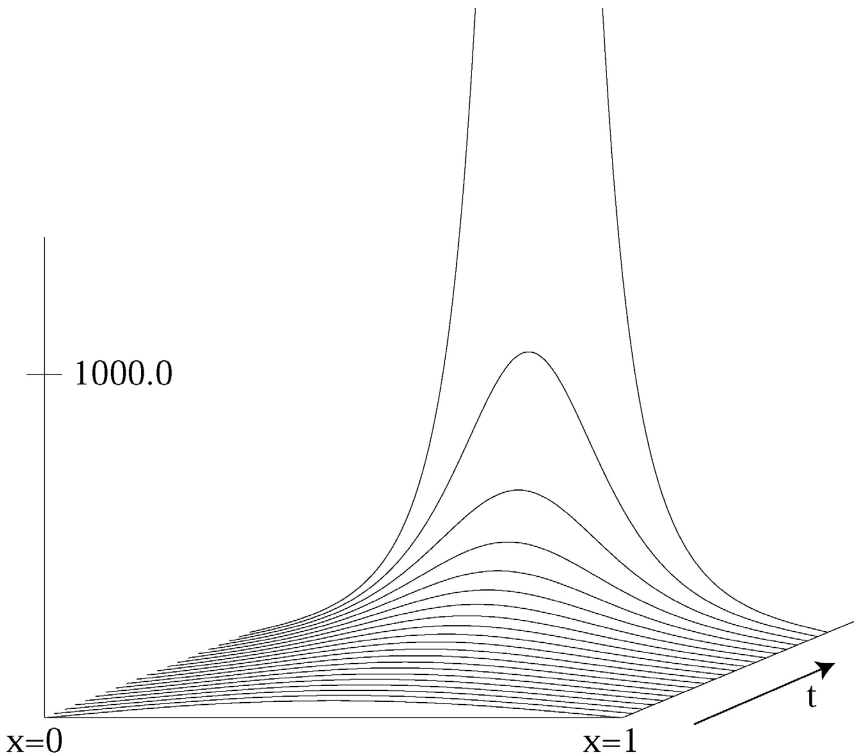
$$\lim_{t \to T} u(t, x_0) = +\infty,$$

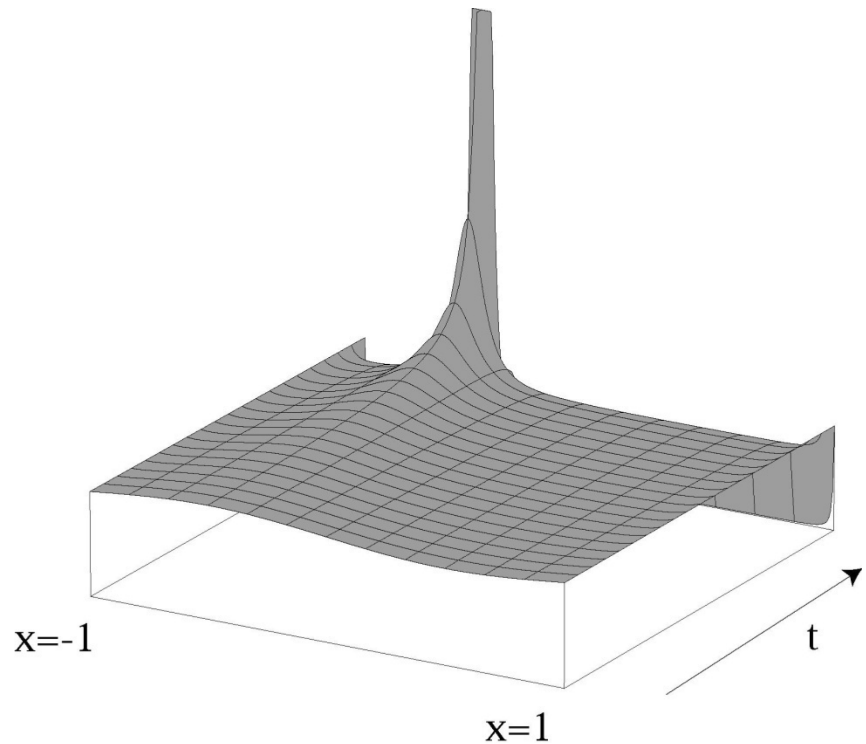
$$\lim_{t \to T} u(t, y) = -\infty \quad (y \neq x_0)$$

$$\lim_{t \to T} \frac{u(t, y)}{u(t, x_0)} = 0$$



Blow-up with or without the projection





$u_t = u_{xx} + u^2$

$u_t = u_{xx} + Pu^2$

Example ④ ; Generalized Proudman-Johnson equation

- A model:

$$f_{txx} + ff_{xxx} - af_x f_{xx} = \nu f_{xxxx}$$

$$(\ 0 < t , \quad -1 < x < 1 \ )$$

$$f(t, \pm 1) = 0, \quad f_x(t, \pm 1) = 0$$

$$f_x(0,x) = \phi(x)$$

$$\omega = -f_{xx}, \quad \omega_t + f\omega_x - af_x\omega = \nu\omega_{xx}$$

Though simple, it contains some known equations as particular members.

❶ a= -(m-3)/(m-1),   axisymmetric exact solutions of the Navier-Stokes equations in $\mathbf{R}^m$.
(Zhu & O. Taiwanese J. Math. 2000) (a=0 for 3D Euler)

❷ a=1 (m=2)   Proudman-Johnson equation ('24, '62)

❸ a=-2, ν=0.   Hunter-Saxton equation ('91)

❹ a=-3   Burgers equation ('40)

$$f_{txx} + f\, f_{xxx} - a f_x f_{xx} = \nu f_{xxxx}$$

Xinfu Chen's proof of global existence

- X. Chen and O., Proc. Japan Acad., vol. 78 (2002),
- periodic boundary condition.

- **THEOREM.** If $-3 \leqq a \leqq 1$, the solutions exist globally in time for all initial data.

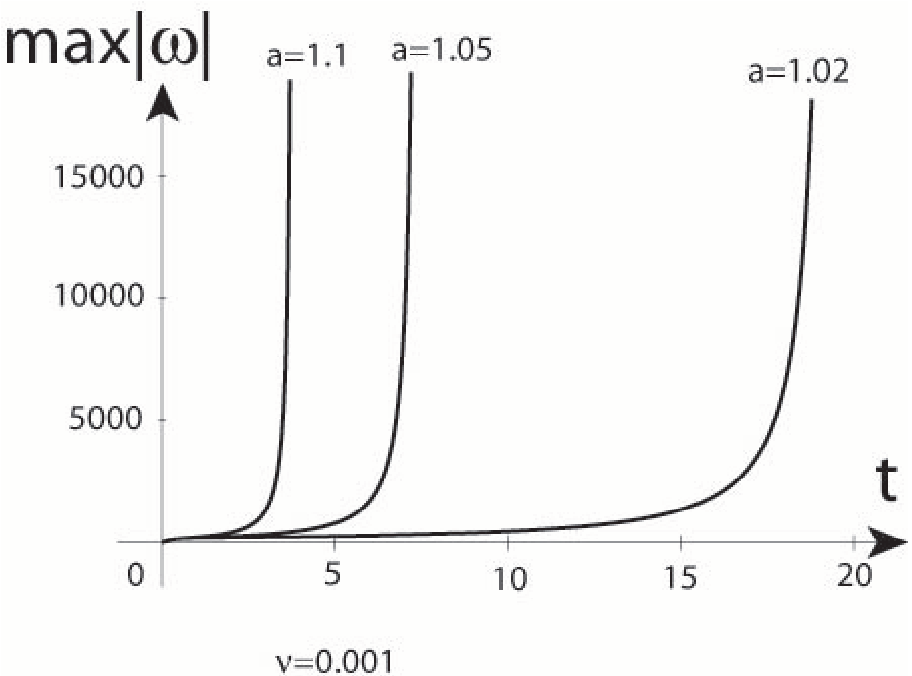
If $a < -3$, or $1 < a$, then …

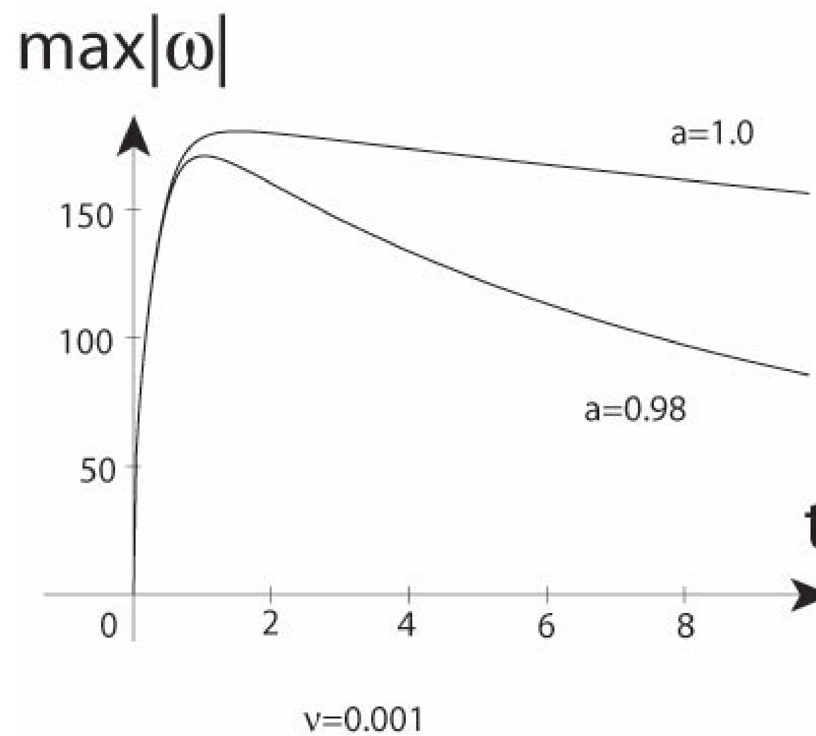
- Global existence for small initial data. Blow-up for large initial data --- numerical evidence but no proof.

- Blow-up sets are $[-1,1]$ for $1 < a$, and discrete for $a < -3$. (no proof)

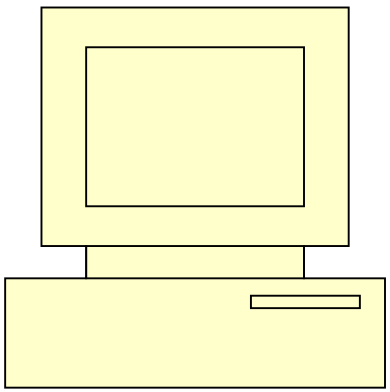
Numerical experiments (Zhu & O. Taiwanese J. Math. 2000)

- a=1 is a threshold.

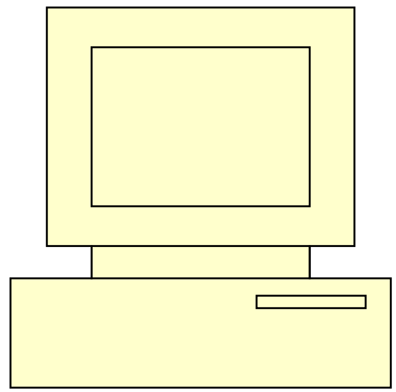




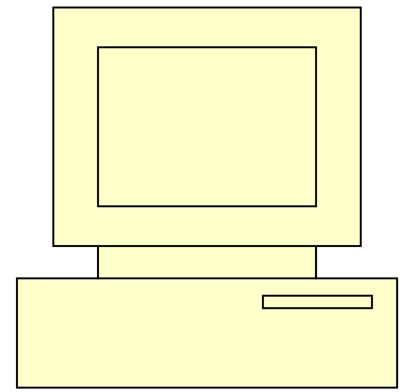
Numerical experiments



$$f_0 = \sin(6\pi x),$$
$$g_0 = 0.2\sin(2\pi x)$$

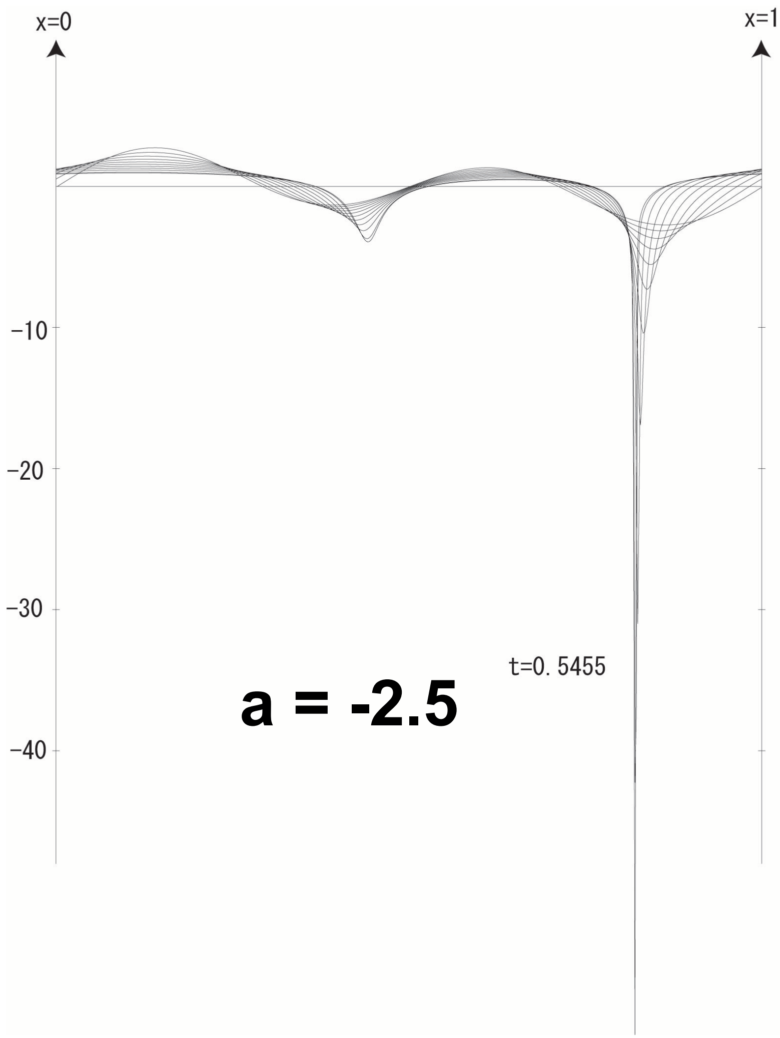


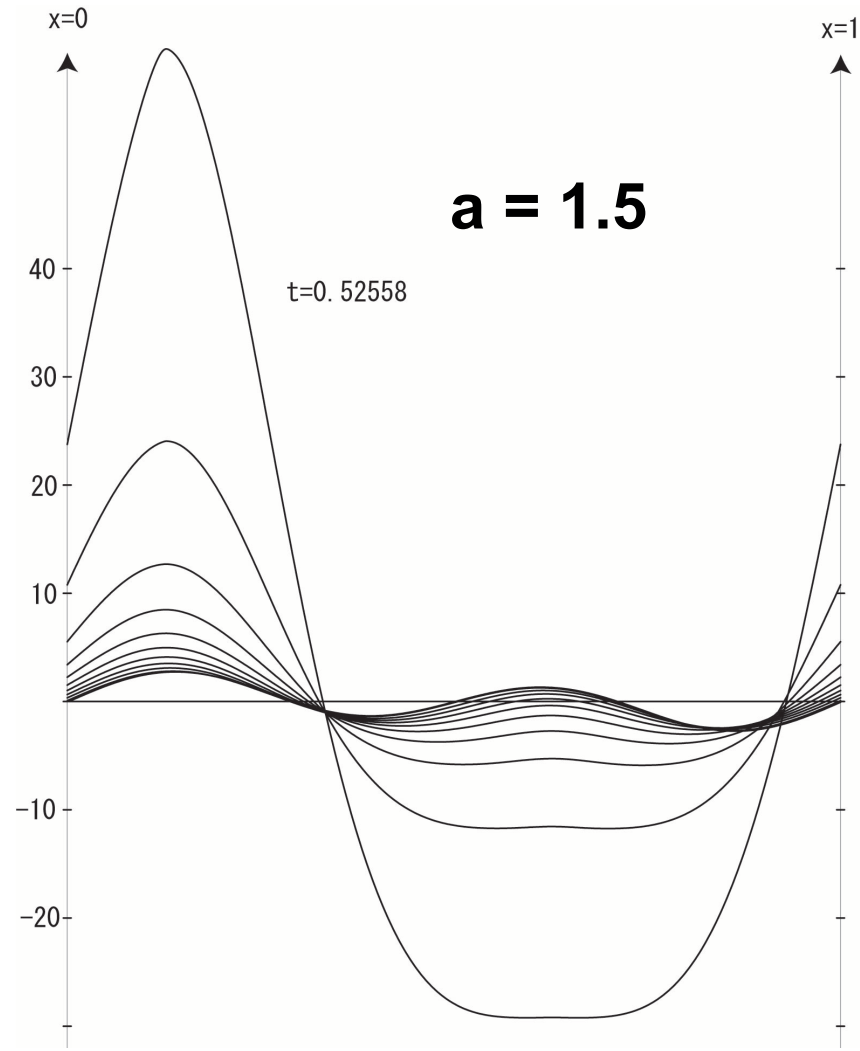
$$f_0 = \sin(10\pi x),$$
$$g_0 = 0.2\cos(2\pi x)$$



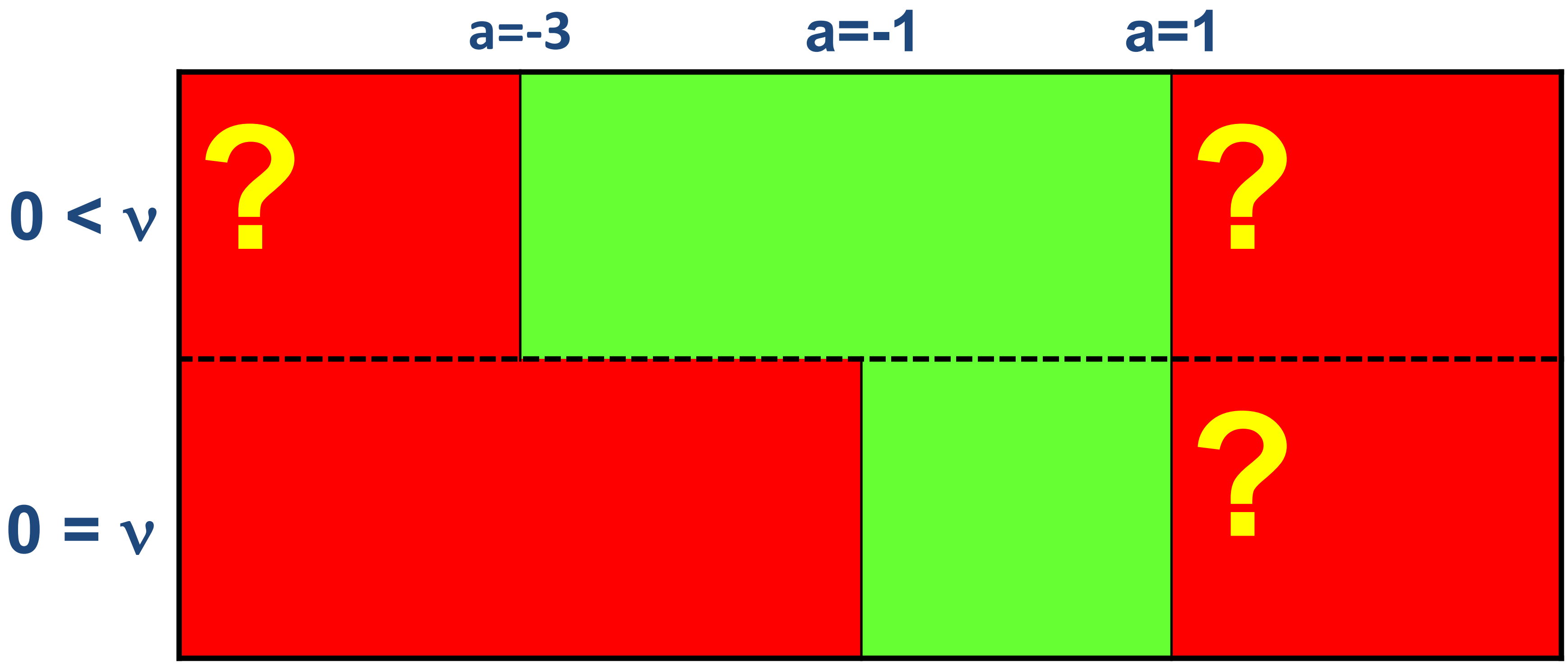
$$f_0 = \sin(16\pi x),$$
$$g_0 = 0.5\cos(8\pi x)$$

If 1 < a, we expect blow-up occurs even for smooth initial data.





Current Status



Conclusion

- Proudman-Johnson eqn’s well-posedness is guaranteed by the convection term.
- Generalized P-J eqn may or may not blow up depending on the parameter a.
- De Gregorio’s equation does not admit blow-up, while Constantin-Lax-Majda eq. admit blow-up.

Conclusion continued

- The regularity of sols. of 2D Euler eqn's seems to be maintained by the convection term;
- Convection term, if properly placed, prevents solutions from blowing up;
- These examples suggests: Blow-up of 3D Navier-Stokes or Euler eqs. is very subtle.

完: Thank you.

Some thoughts on the role of the convection term in the fluid mechanical PDEs.

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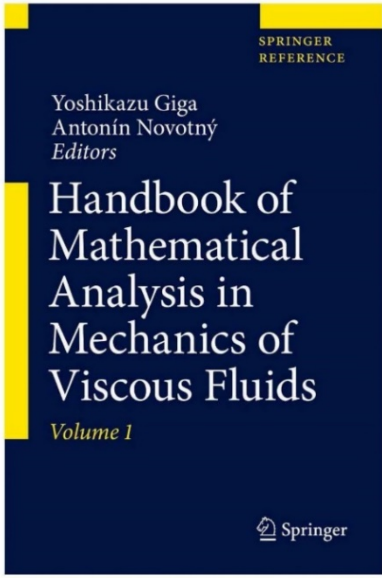
Today's goal

- Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Editors: Giga & Novotny to appear in Springer in 2018. Includes: O., Models and special solutions of the Navier-Stokes equations, in Handbook



- Bae, Chae & O., Nonlinear Analysis (2017)
- Ohkitani & O., J. Phys. Soc. Japan, (2005)
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- H. O., *J. Math. Fluid Mech.* Online (2007)
- K. Ohkitani and H.O., *J. Phys. Soc. Japan*, 74 (2005), 2737--2742.
- H.O. & J. Zhu, *Taiwanese J. Math.*, 4 (2000), 65—103

A motive: 3D Navier-Stokes: A bad problem. Turbulence is a bad Problem!? How about the NS itself?

Try simpler models for blow-up:

- **Proudman-Johnson eq. (special sol.)**
- **Constantin-Lax-Majda (model)**
- **generalized CLM (model)**
- **Surface QG, & many others.**
- **Model equations for water waves.**

Navier-Stokes is nonlinear & nonlocal

- Navier-Stokes eqns. are *integro-differential* eqns. rather than differential eqns.

$$\boldsymbol{\omega}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \nabla)\mathbf{u} = \nu \Delta \boldsymbol{\omega}$$

convection **stretching** **viscosity**

$$\mathbf{u} = (\mathrm{curl})^{-1}\boldsymbol{\omega}, \quad \mathrm{Biot-Savart}$$

$$\mathbf{u}(t,x) = \frac{-1}{4\pi}\iiint \frac{x-\xi}{|x-\xi|^3} \times \boldsymbol{\omega}(t,\xi)\, d\xi$$

Therefore models must be nonlinear & nonlocal.

When I began my career as a professional mathematician, there was a *folklore*:

- The vorticity is increased by the stretching term. Convection term does not increase vorticity, although the vorticity is rearranged by that.
- As far as global well-posedness is concerned, the convection term is neutral.

Can these loose ``propositions'' be phrased mathematically?

The Proudman-Johnson equation. '62

$$u = u(t, x) : \text{unknown}$$

$$u_{txx} + uu_{xxx} - u_x u_{xx} = \nu u_{xxxx}$$

$$(0 < t,\ 0 < x < 2\pi)$$

periodic BC

Hiemenz's ansatz

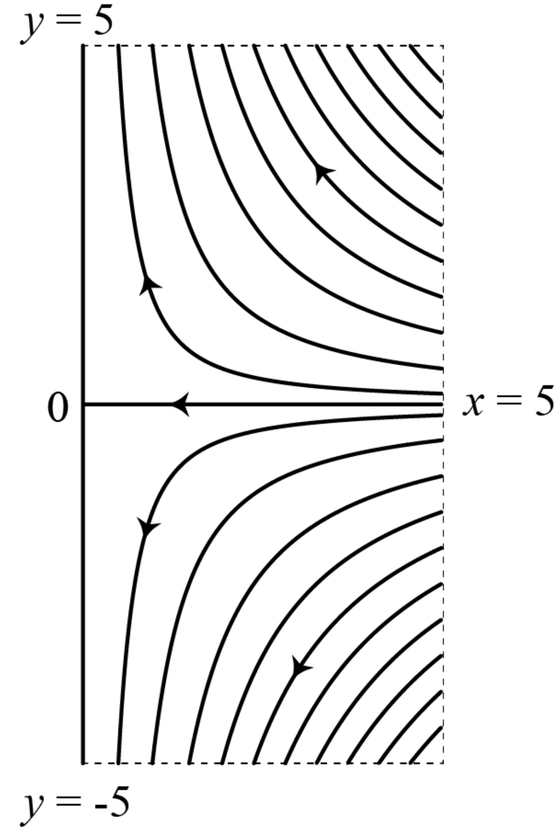
Dinglers Journal 1911

Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder

He obtained a steady-state.

$$\mathbf{u} = (u(x), -y u_x(x)) \quad \Rightarrow \quad \operatorname{div} \mathbf{u} = 0$$

$$u u_{xxx} - u_x u_{xx} = \nu u_{xxxx}$$



DINGLERS
POLYTECHNISCHES JOURNAL.
Herausgeber: Geheimer Regierungsrat Professor M. Rudeloff, Groß-Lichterfelde-West.

Jährlich 52 Hefte in Quart. Abonnementspreis vierteljährlich 6 M., direkt franko unter Kreuzband für Deutschland und Oesterreich-Ungarn 6 M. 65 Pf., für das Ausland 7 M. 30 Pf. Redaktionelle Sendungen sind zu richten an Geh. Reg.-Rat Prof. M. Rudeloff, Groß-Lichterfelde-West, Fontanestraße, die Expedition betreffende Schreiben an Richard Dietze, Verlagsbuchhandlung (Dr. R. Dietze), Berlin W. 66, Mauerstraße 15.

92. Jahrg., Bd. 326. | Berlin, 27. Mai 1911. | Heft 21.

Die Grenzschicht an einem in den gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder.

Von K. Hiemenz.

Einleitung.

Bei der Behandlung der Strömungserscheinungen um ein in den Flüssigkeitsstrom eingestelltes Hindernis sieht der gewöhnliche Ansatz der Hydrodynamik ab von der inneren Reibung der Flüssigkeiten und führt so zu einer verhältnismäßig einfachen Lösung des Problems: die Geschwindigkeitskomponenten lassen sich mit Hilfe einer Potentialfunktion darstellen. Wird die innere Reibung berücksichtigt, so lauten die Differentialgleichungen der stationären Strömung für das zweidimensionale Problem, von welchem im folgenden ausschließlich die Rede sein soll:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+k\,\Delta u \quad . \quad . \quad 1a)$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+k\,\Delta v \quad . \quad . \quad 1b)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 1c)$$

Hierin bedeuten: x, y die gewöhnlichen rechtwinkligen Koordinaten, u, v die Geschwindigkeitskomponenten in Richtung der Achsen, Δ den *Laplace*schen Operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Eine mögliche Lösung des Systems 1 von Differentialgleichungen hat man in der Potentialströmung für das betreffende Problem. Aber diese Lösung steht im Widerspruch zu der Grenzbedingung der reibenden Flüssigkeit, wie sie sich auf Grund experimenteller Tatsachen ergibt; der Versuch führt zu der Folgerung, daß die an die Wand grenzende Flüssigkeitsschicht an dieser haftet. Im Gegensatz dazu treten bei der Potentialbewegung gerade unmittelbar an der Wand die größten Geschwindigkeiten auf, und daher ist die Potentiallösung für reibende Flüssigkeiten nicht brauchbar. Eine selbst äußerst geringe Reibung bedingt eine wesentliche Abweichung der im einzelnen Falle zu erwartenden Strömungserscheinung von der für das gleiche Problem gefundenen Potentialbewegung.

Bis jetzt ist die Integration des Systems 1 nur in speziellen Fällen durchgeführt worden. Dazu gehören einerseits gewisse einfache Laminarbewegungen, andererseits eine Reihe von Problemen, bei denen die konvektiven Glieder der Differentialgleichungen gegenüber den anderen Gliedern vernachlässigt wurden. Die praktische Anwendbarkeit der unter dieser Bedingung gefundenen Integrale ist beschränkt. Denn bei wirklichen Flüssigkeiten — bei Wasser ist $\rho = 1$, $k \sim 0{,}01$ $c-g-s$-Einheiten — sind in der Regel die auftretenden Geschwindigkeiten nicht klein genug, um die Vernachlässigung der konvektiven Glieder zu erlauben. Dagegen gestattet eine nach anderer Richtung gehende von *Prandtl*[1]) angegebene Vereinfachung des Systems 1, die sich auf Flüssigkeiten von kleiner Reibung bezieht, die Strömungsvorgänge in der Nähe der festen Wand in ihrem Verlaufe zu verfolgen. Der *Prandtl*sche Ansatz ist weiter ausgebaut worden für eine Reihe von Problemen der stationären und der nichtstationären Strömung in zwei Arbeiten von *Blasius*[2]) und von *Boltze*[3]). Die Resultate dieser Arbeiten geben ein sehr gutes qualitatives Bild der beobachteten Vorgänge der Strömung um ein Hindernis.

Die vorliegende Arbeit hat demgegenüber als Endziel die quantitative Prüfung des *Prandtl*schen Ansatzes durch das Experiment. Demgemäß wird in folgendem nach einem einleitenden mathematischen Teil von Experimenten berichtet werden, die auf eine quantitative Kenntnis der Strömungserscheinungen an einem Hindernis — in erster Linie der Druckverteilung — hinzielen. Die experimentell ermittelten Werte werden sodann zur Grundlage der Rechnung gemacht werden, und schließlich sollen die errechneten Strömungserscheinungen mit den wirklich beobachteten verglichen werden. Als Endresultat ergibt sich eine durchaus befriedigende Uebereinstimmung von Rechnung und Versuch.

I. Die Differentialgleichung der Grenzschicht.

1. *Ableitung der Gleichung.*

Wir beginnen mit einer qualitativen Schilderung der in einer Flüssigkeit von kleiner Reibung beobachteten Strömungserscheinungen. Die mathematische Präzisierung der Ergebnisse der Beobachtung führt zu der bereits erwähnten Vereinfachung des Systems 1. Im Interesse der Kürze beziehen wir uns dabei von vornherein auf den Fall, der uns weiterhin beschäftigen wird: In den gleichförmigen Strom einer Flüssigkeit von kleiner Reibung sei ein symmetrischer gerader

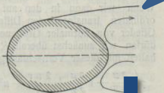


Fig. 1.

[1]) *Prandtl.* Ueber Flüssigkeitsbewegung bei sehr kleiner Reibung. Verhandlungen des dritten internationalen Mathematikerkongresses in Heidelberg 1904. Leipzig 1905, S. 484.

[2]) *Blasius,* Grenzschichten in Flüssigkeiten mit kleiner Reibung. Göttinger Diss. Leipzig 1907. Auch in Zeitschrift f. Math. u. Phys. Bd. 55. Leipzig 1908, S. 1.

[3]) *Boltze,* Grenzschichten an Rotationskörpern i. Flüssigkeiten mit kleiner Reibung. Diss. Göttingen 1908.

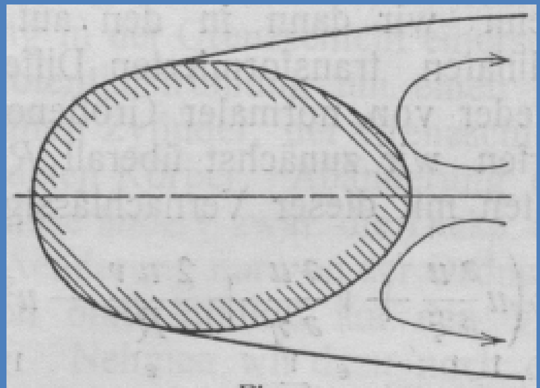


Fig. 1.

Laminar boundary layer

The Proudman-Johnson equation. '62

- Derived from 2D Navier-Stokes

  (unbounded solution of NS)

$$\mathbf{u} = (u(t,x), -y u_x(t,x))$$

$$u_{txx} + u u_{xxx} - u_x u_{xx} = \nu\, u_{xxxx}$$

$(0 < t,\ 0 < x < 2\pi)$

periodic BC & $u_{xx}(0, x) = -\phi(x)$ IC

Equivalent re-writing

$$u_{txx} + u u_{xxx} - u_x u_{xx} = \nu u_{xxxx}$$

$$\omega = -u_{xx}$$

$$\omega_t + u\omega_x - u_x \omega = \nu \omega_{xx}$$

$$\boldsymbol{\omega}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \nabla)\mathbf{u} = \nu \Delta \boldsymbol{\omega} \quad \& \ \text{Biot-Savart}$$

Generalized Proudman-Johnson equation

Zhu & O. Taiwanese J. Math., vol. 4 (2000),

A model:

$$\omega_t + u\omega_x - au_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

$$\omega(0,x) = \omega_0(x)$$

1. $a = -(m-3)/(m-1)$, **axisymmetric exact solutions of the Navier-Stokes eqns in** R^m.
2. $a = 1$ $(m=2)$ **Proudman-Johnson eqn**
3. $a = -2$, $\nu = 0$. **Hunter-Saxton equation ('91)**
4. $a = -3$ **the Burgers equation ('46)**

$$\frac{d^2}{dx^2} \quad u_t + uu_x = \nu u_{xx} \quad \Rightarrow \quad u_{txx} + uu_{xxx} + 3u_x u_{xx} = \nu u_{xxxx}$$

Global existence or finite time blow-up?

$$u_{txx} + uu_{xxx} - u_x u_{xx} = \nu u_{xxxx}$$

$$\omega = -u_{xx}$$

Order -2

$$\omega_t + u\omega_x - u_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

$$\omega(0,x) = \omega_0(x)$$

$$\boldsymbol{\omega}_t + (\mathbf{u} \bullet \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \bullet \nabla)\mathbf{u} = \nu\Delta\boldsymbol{\omega}$$

$$\mathbf{u} = (\mathrm{curl})^{-1}\boldsymbol{\omega}, \quad \text{Biot} - \text{Savart}$$

Order -1

In 1989, a paper appeared in *J. Fluid Mech*.

- Finite time blow-up was predicted by numerical computation.
- For ten years, I was wondering if that is true.

$$u_{txx} + uu_{xxx} - u_x u_{xx} = \nu u_{xxxx}$$

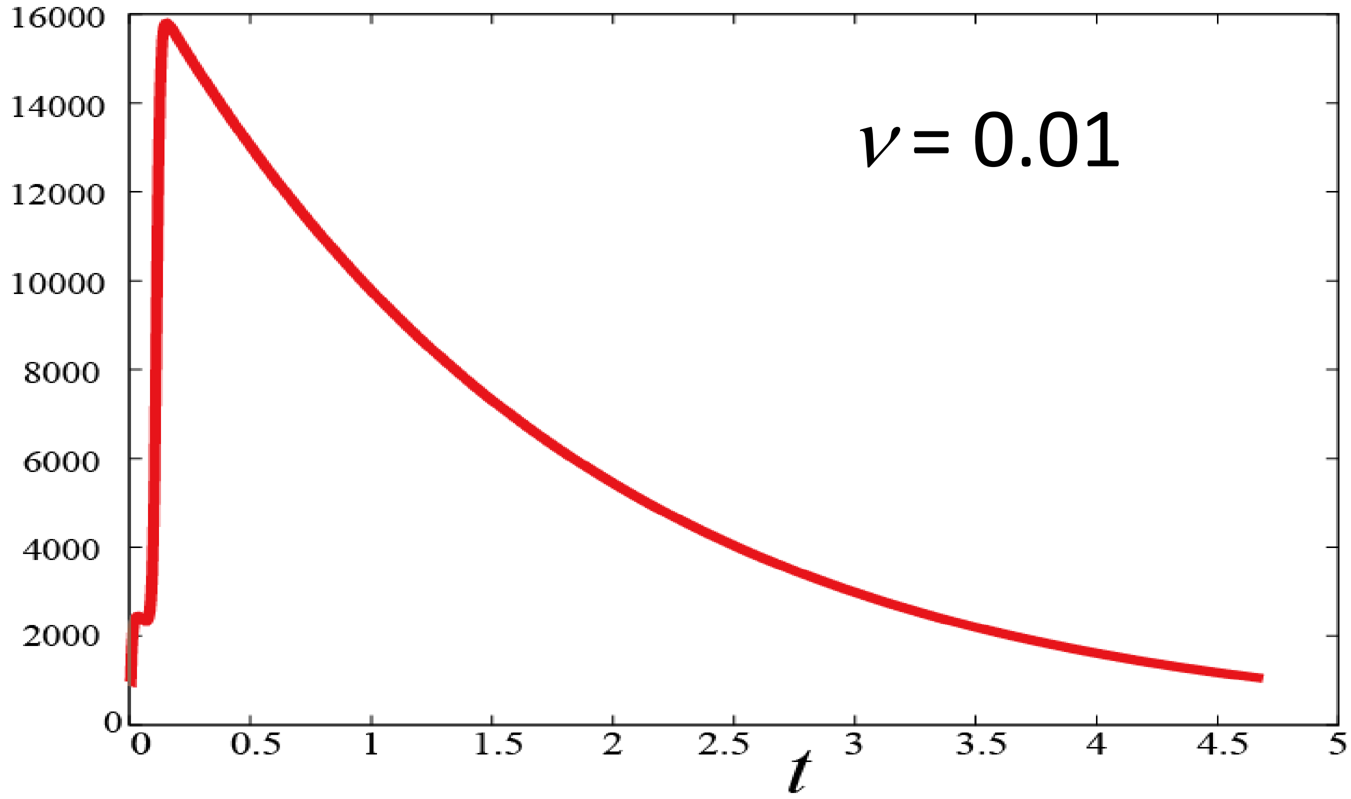
$$u_{tx} + uu_{xx} - (u_x)^2 = \nu u_{xxx} + \gamma(t)$$

$$w_t + uw_x = \nu w_{xx} + w^2 + \gamma(t)$$

$$w = u_x$$

Nonlinear heat equation with nonlocal nonlinear convection

Max norm of u_{xx} $\quad \|u_{xx}(t,\bullet)\|_{L^\infty}$



Global existence was proved for PJ:

Theorem. Assume that viscosity $\nu > 0$. For any initial data in $L^2(-1,1)$, a solution exists uniquely for all t and tends to zero as $t \to \infty$.

if homogeneous Dirichlet, Neumann, or the periodic boundary condition.

Xinfu Chen and O., Proc. Japan Acad., 2000.

Blow-up if non-homogeneous Dirichlet BC.????

$u(t,1) = a,\ u_{xx}(t,1) = b,\ u(t,-1) = c,\ u_{xx}(t,-1) = d$

Grundy & McLaughlin (1997).

proof: a priori estimate

$$u_{txx} + uu_{xxx} - u_x u_{xx} = \nu u_{xxxx}$$

$$u_{txxx} + uu_{xxxx} - \left(u_{xx}\right)^2 = \nu u^{(V)}$$

$$u_{txxx} + uu_{xxxx} \geq \nu u^{(V)}$$

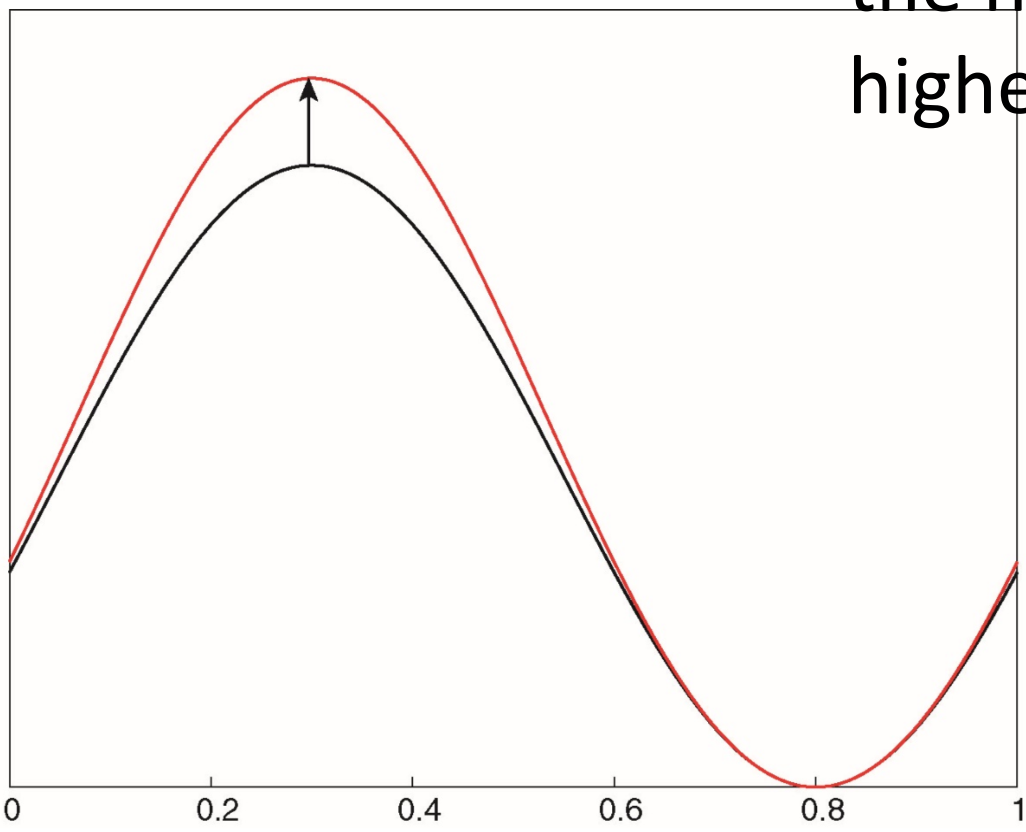
$$\zeta_t + u\zeta_x \geq \nu \zeta_{xx}$$

$$\int_0^{2\pi} u_{xxx}(t,x)dx \equiv 0$$

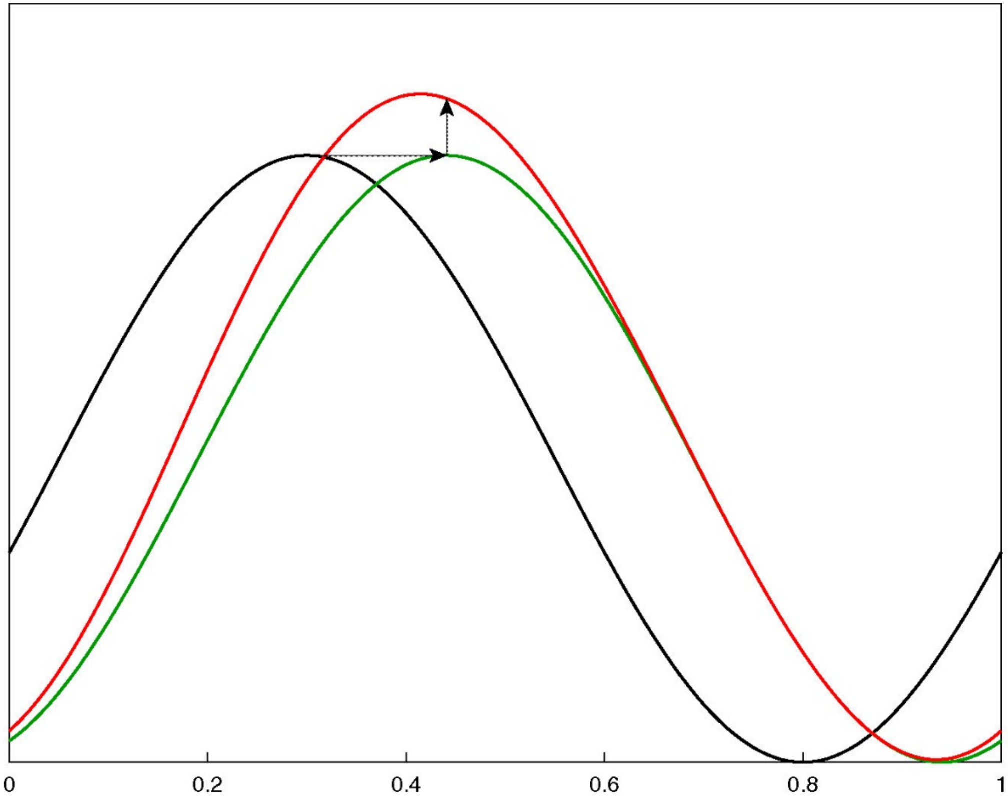
Maximum principle

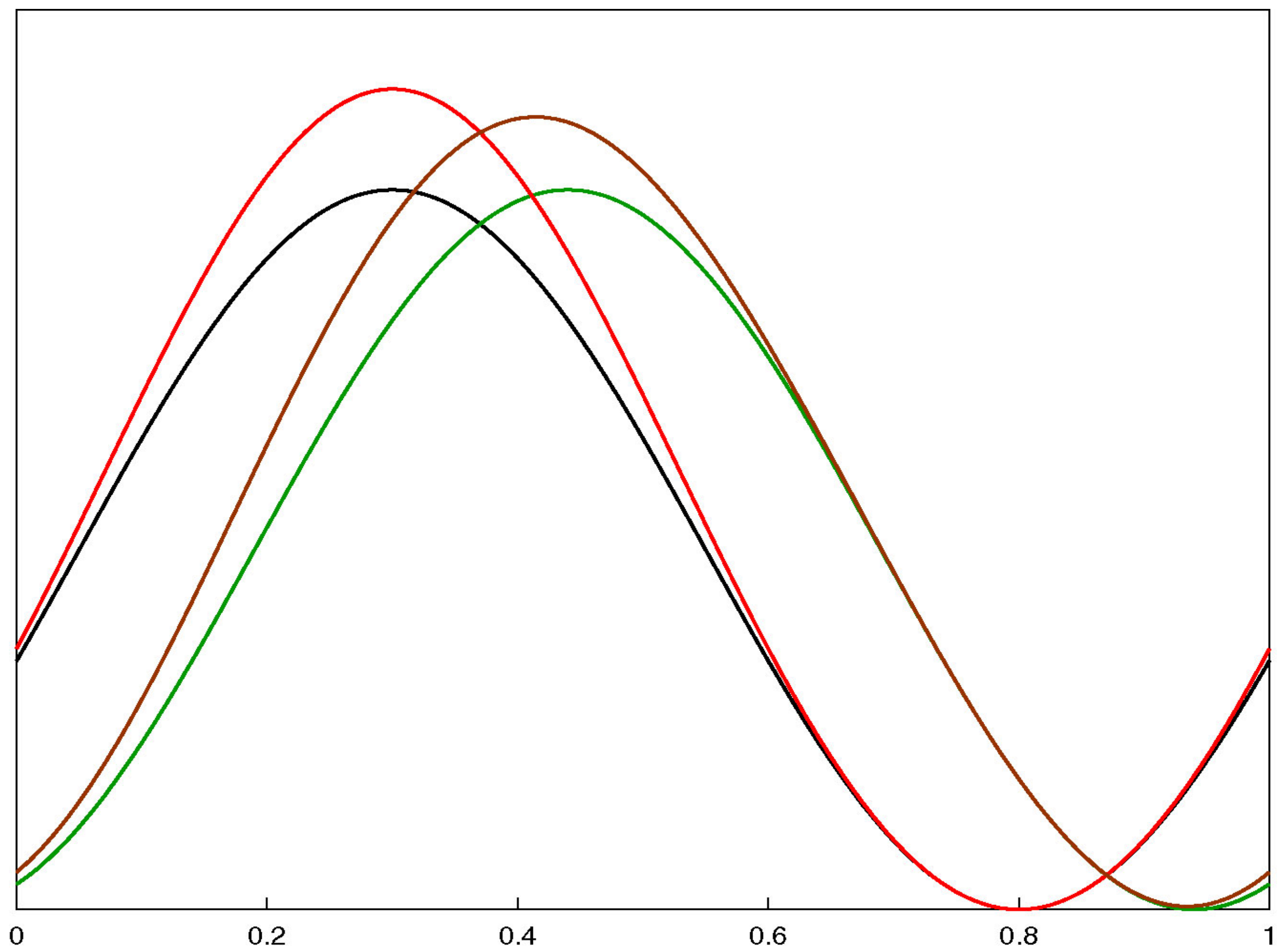
Intuitive explanation

- Without convection, the higher becomes higher still.

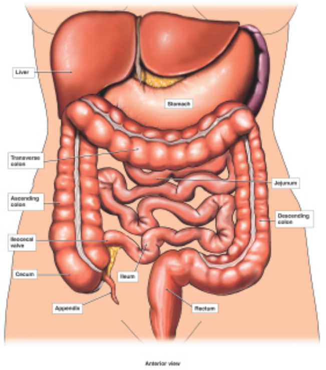


- With convection, the highest does not necessarily become the highest.



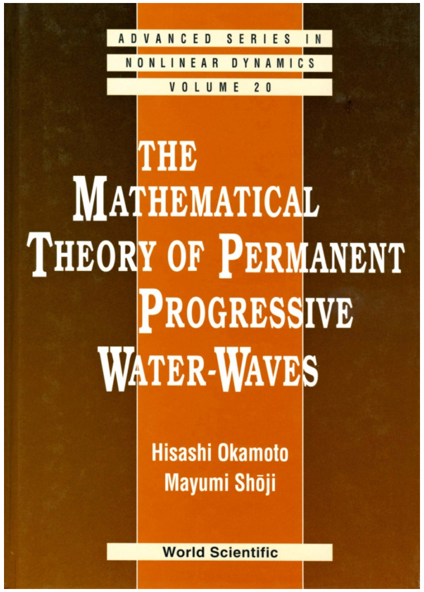

0
0.2
0.4
0.6
0.8
1

Surgery on PDE



- If appendix is removed, we can live

  ⇒ Appendix is unnecessary as far as life/death is concerned.



- Is the convection term an appendix, or not?

Surgery on convection term

$$u_t + uu_x = \nu u_{xx}$$

$$u_{tx} + uu_{xx} + (u_x)^2 = \nu u_{xxx}$$

$$w_t + uw_x + w^2 = \nu w_{xx} \qquad w = u_x$$

- Remove

$$w_t + w^2 = \nu w_{xx}$$

$$w_t - uw_x + w^2 = \nu w_{xx}$$

Surgery on convection term

$$\omega_t + u\omega_x - u_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

$$u_{txx} - u_x u_{xx} = \nu u_{xxxx}$$

$$u_{tx} - \frac{1}{2}u_x^{\ 2} = \nu u_{xxx}$$

$$U = \frac{1}{2}u_x, \quad \mathrm{U}_t = \nu U_{xx} + U^2 - b(t)$$

$$U_t = U_{xx} + U^2 - \frac{1}{2}\int_{-1}^{1} U(t,x)^2\,dx, \quad (0 < t, -1 < x < 1)$$

$$\int_{-1}^{1} U(t,x)\,dx = 0, \quad \text{periodic BC}$$

Blow-up occurs

$$U_t = U_{xx} + PU^2, \quad P : L^2 \to L^2/\mathbf{R}$$

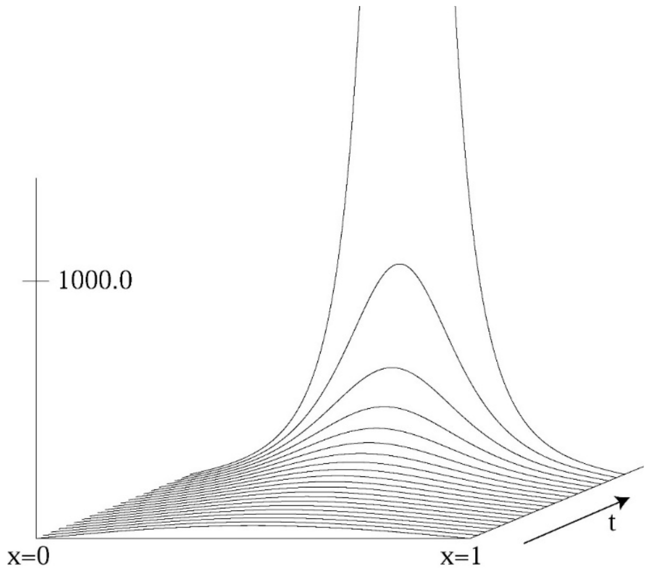
$$\omega_{txx} + u\omega_x - u_x\omega = \nu\omega_{xx},$$ ← **Global existence**

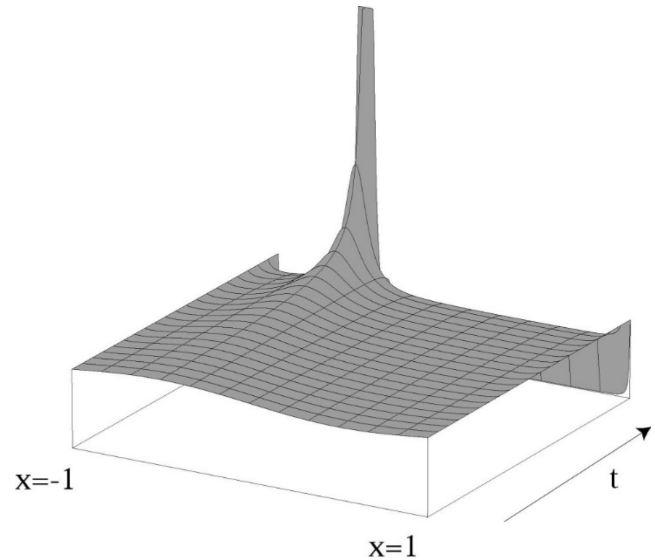
$$\omega_{txx} \qquad - u_x\omega = \nu\omega_{xx}$$ ← **Blow-up**

A proper convection term prevents solutions from blowing-up.

(O. & J. Zhu 1999, Taiwanese J. Math., 2000
Cf. Budd, Dold & Stuart ('93),)

Blow-up with or without the projection





$$u_t = u_{xx} + u^2 \qquad\qquad u_t = u_{xx} + \mathbf{P}u^2$$

Generalized Proudman-Johnson equation

A model

$$\omega_{txx} + u\omega_x - au_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

$$\omega(0,x) = \phi(x)$$

❶ a = -(m-3)/(m-1), **axisymmetric exact solutions of the Navier-Stokes eqns in R^m.**

❷ a = 1 (m=2) **Proudman-Johnson eqn**

❸ a = -2, ν=0. **Hunter-Saxton equation ('91)**

❹ a = -3 **the Burgers equation ('46)**

$$\frac{d^2}{dx^2} \quad u_t + uu_x = \nu u_{xx} \quad \Rightarrow \quad u_{txx} + uu_{xxx} + 3u_x u_{xx} = \nu u_{xxxx}$$

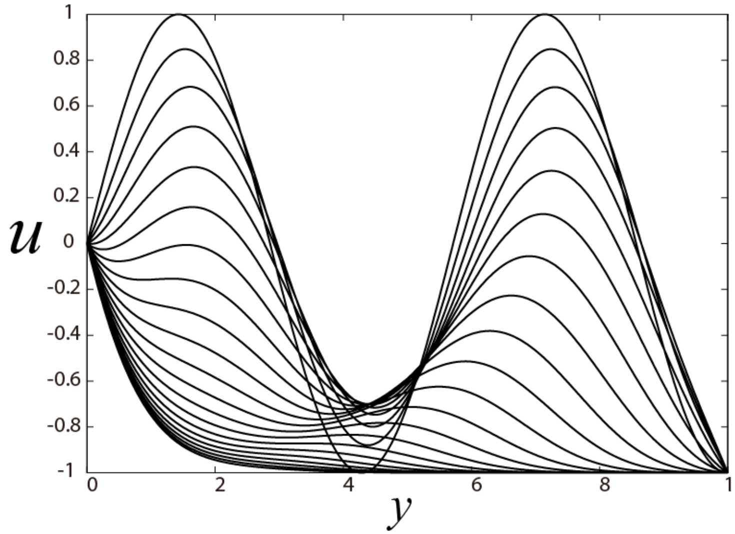
No singularity if the convection term is dominant.

- If a is large $\Rightarrow$ blow-up
- If a is small $\Rightarrow$ no blow-up

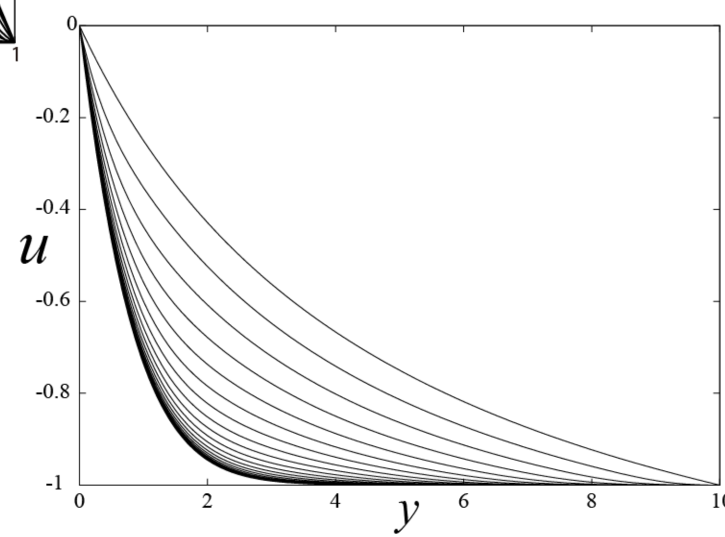
$$\omega_{txx} + u\omega_x - au_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

But how large it must be?

Converges to Hiemenz's steady-state



$$u(t,\infty) = -1$$



Conjecture

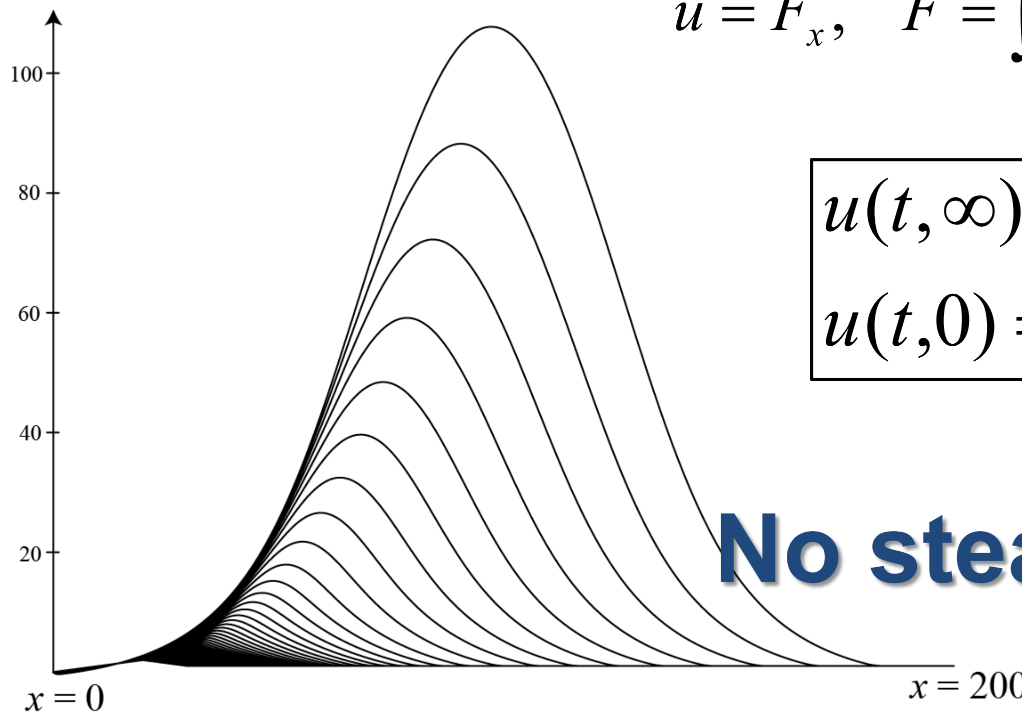
- If $u_0 \leqq 1$ everywhere, global.
- If $u_0 > 1$ somewhere, blow-up?
- $u = F_x$ is bounded, but F is unbounded.

$$u_t + F u_x - u^2 = \nu u_{xx} - 1$$

$$u_t + F u_x = \nu u_{xx} + u^2 - 1$$

$$u = F_x, \quad F = \int_0^x u$$

$$u(t,\infty) = 1$$
$$u(t,0) = 0$$



No steady-state

Unimodal conjecture on the $\boldsymbol{PJ}$

Kim & Okamoto, IMA J. Appl. Math. (2013)

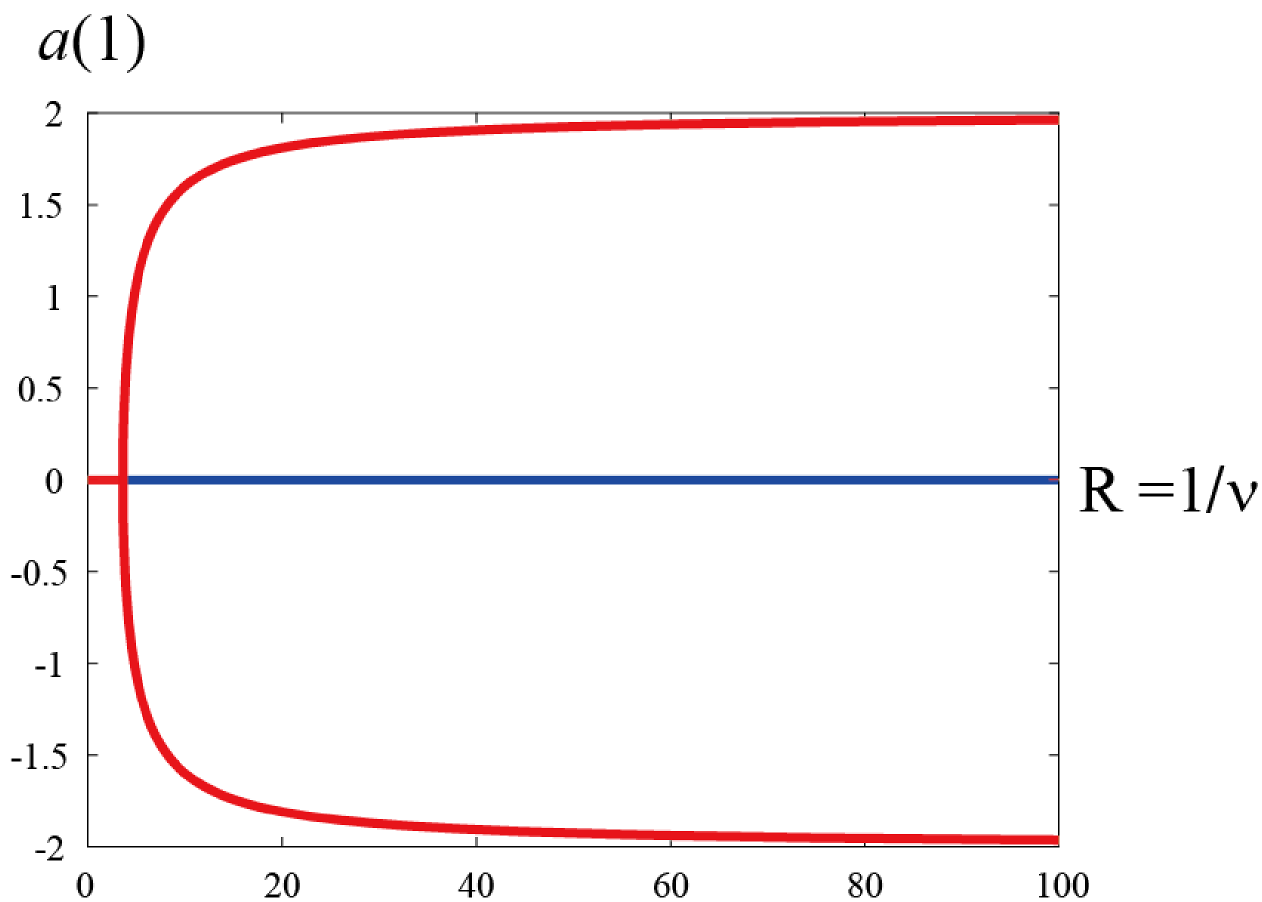
$$F_{txx} + FF_{xxx} - F_x F_{xx} = \nu\left(F_{xxxx} - \sin kx\right)$$

$$-\pi < x < \pi, \quad \text{Periodic BC}$$

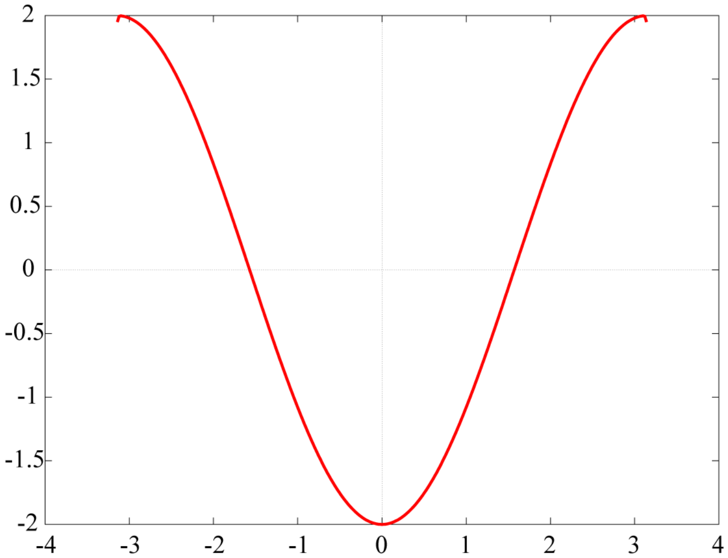
$$F = k^{-4} \sin kx$$

$$\forall\, k = 1,2,3,\ldots$$

$$\exists\ \text{sol} \quad F = k \sin x \quad \textit{for} \quad 0 < \nu << 1$$



a(1)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
20
40
60
80
100
R =1/ν

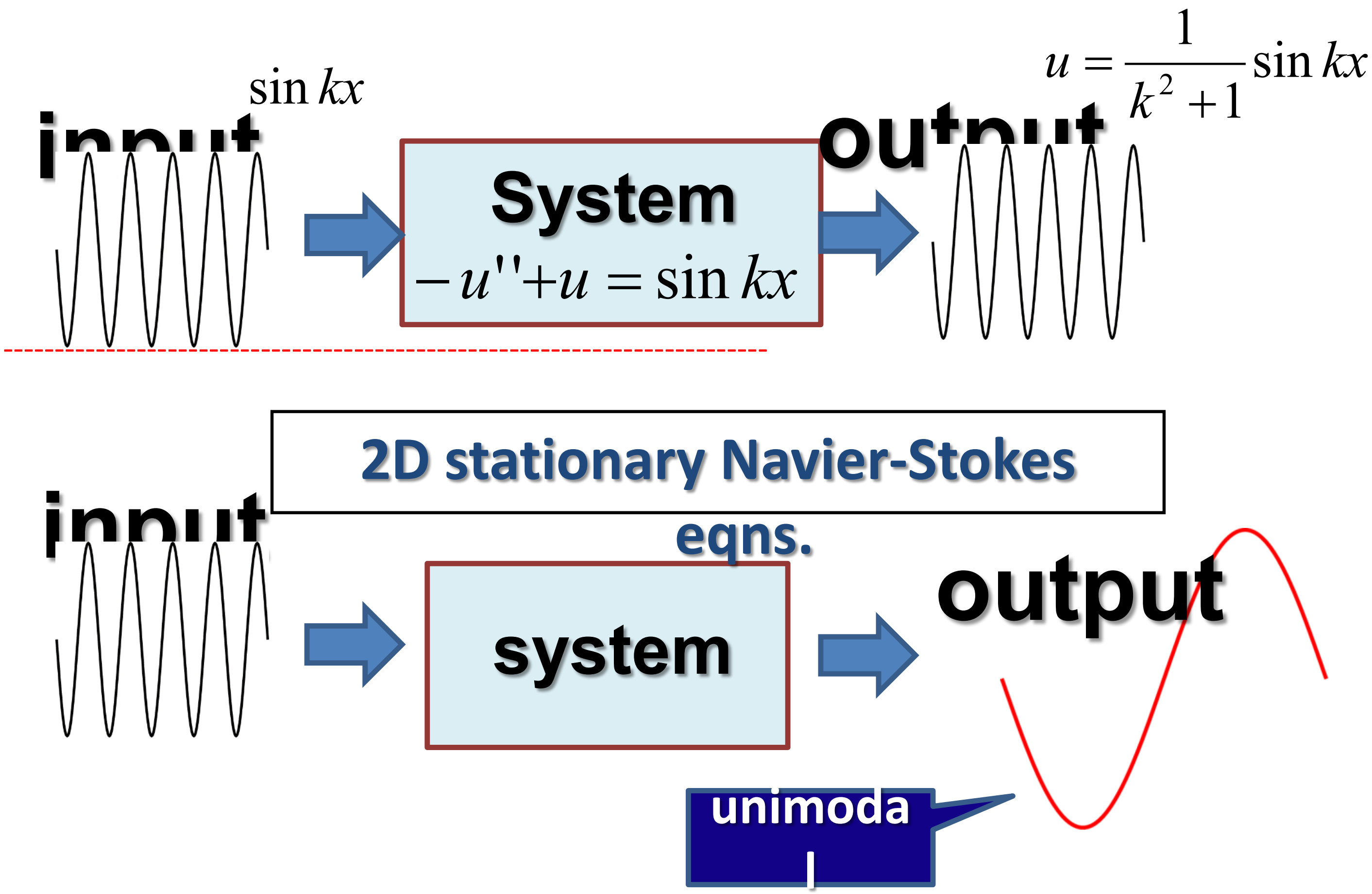


$$R = 5000$$

$a(1)$	−1.999270373875
$a(2)$	0.000044444742
$a(3)$	0.000006247917
$a(4)$	0.000001776474
$a(5)$	0.000000693595
$a(6)$	0.000000325941
$a(7)$	0.000000173180
$a(8)$	0.000000100453

$$\psi(x) = \sum_{n=1}^{N} a(n) \sin nx$$

Linear & weakly nonlinear eqns.



Theorem
There exists a sol. of the fol. form.

$$\psi(x) = \pm k \sin x + R^{-1} h(x) + O(R^{-2}) \quad (R \to \infty)$$

$$\psi_{txx} + \psi \psi_{xxx} - \psi_x \psi_{xx} = \frac{1}{R}\left(\psi_{xxxx} + \sin kx\right)$$

$$h(x) = c_1 \sin x + \frac{2}{9} \sin 2x + 0 \times \sin 3x + \frac{2}{225} \sin 4x$$

$$\frac{2}{9} = 0.2222\ldots, \qquad \frac{2}{225} = 0.008888\ldots \quad \boxed{\frac{2}{9R}}$$

	k = 4, R = 10000
a_1	3.99887570831577
a_2	0.00002222222222
a_3	0.00000000054426
a_4	0.00000088889289

3D flows

- C.C. Lin, Arch. Rat. Mech. Anal. (1957)
- Grundy & McLaughlin, IMA J. Appl. Math. (‘99)
- J. Zhu, Japan J. Indust. Appl. Math. (2000)
- Ansatz

$$\mathbf{u} = \big(f(t,x) - g(t,x),\, -yf_x(t,x),\, zg_x(t,x)\big)$$

$$f_{txx} + (f - g)f_{xxx} - (f_x + g_x)f_{xx} = \nu\, f_{xxxx}$$
$$g_{txx} + (f - g)g_{xxx} + (f_x + g_x)g_{xx} = \nu\, g_{xxxx}$$

Blow-up for 3D

Stagnation-point Flows of Incompressible Fluid

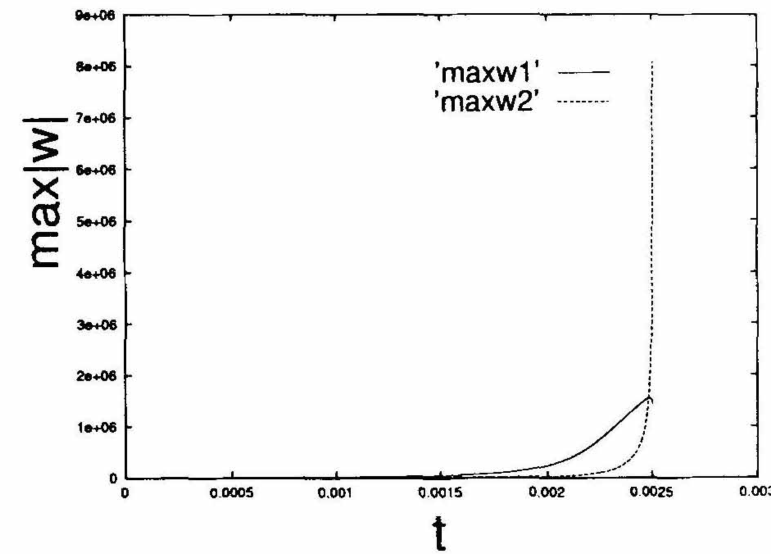


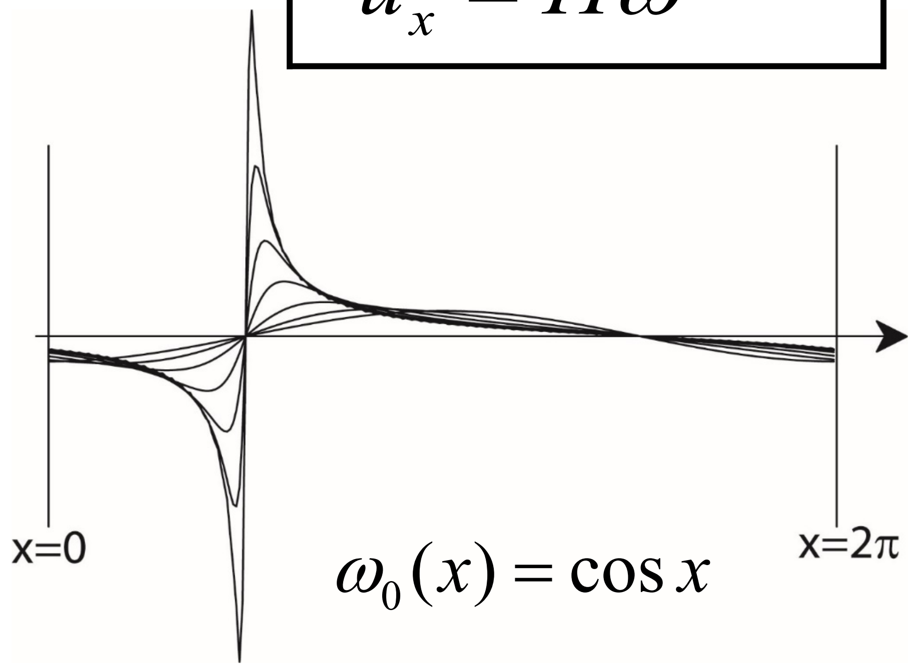
Fig. 17. The same as in Figures 15 and 16 except for $q = \tilde{0}.0001, N = 600$

(No blow-up for PJ)

model ③
Constantin-Lax-Majda

$$\omega_t - \omega u_x = 0$$
$$u_x = H\omega$$

A necessary and sufficient condition is known (Constantin, Lax, & Majda 1985).



Constantin-Lax-Majda & De Gregorio & Proudman-Johnson

$$\omega_{txx} + u\omega_x - au_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-1}\omega$$

$$\omega(0,x) = \phi(x)$$

$$\omega_{txx} + u\omega_x - au_x\omega = \nu\omega_{xx}, \quad u = \left(-\frac{d^2}{dx^2}\right)^{-\beta/2}\omega$$

$$\omega(0,x) = \phi(x)$$

$\beta = 1$ & $a = \infty$ ⟹ **Blow-up Constantin-Lax-Majda `85**

$\beta = 1$ & $a = 1$ ⟹ **??? De Gregorio's `90**

$\beta = 1$ & $-\infty < a < 0$ ⟹ **Blow-up Castro & Cordoba '09**

The generalized P-J with ν=0.

$$u_{txx} + uu_{xxx} - au_x u_{xx} = 0$$

$$(0 < t,\ 0 < x < 1)$$

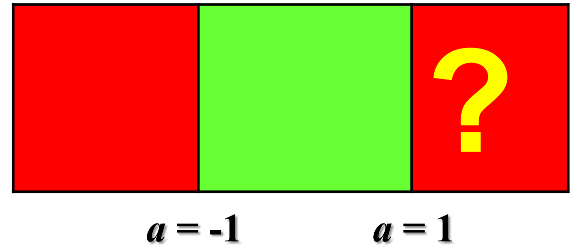
periodic BC

$$u_{xx}(0, x) = -\phi(x)$$

- **3D axisymmetric Euler for $a = 0$.**
- **Hunter-Saxton model for nematic liquid crystal for a = -2.**
- **Burgers for a = -3.**

$$\frac{d^2}{dx^2} \quad u_t + uu_x = 0 \ \Rightarrow u_{txx} + uu_{xxx} + 3u_x u_{xx} = 0$$

Summary for $\nu = 0$



- Blow-up for $-\infty < a < -1$. (Remember that the solutions exist globally in this region if $\nu > 0$. Viscosity helps global existence.)
- Global existence if $-1 \leqq a < 1$ & if smooth initial data.
- Self-similar, non-smooth blow-up solutions exist for $-1 < a < \infty$.
- So far, I have no conclusion in the case of 1 < a.

Starting point: local existence theorem

- With a help of Kato & Lai's theorem (J. Func. Anal. '84),

$$\omega = -u_{xx}, \quad \omega_t + u\omega_x - au_x\omega = 0$$

- Locally well-posed if $\omega(0,\bullet) \in L^2(0,1)/\mathbf{R},$
- Global existence if $\omega(0,\bullet) \in L^2(0,1)/\mathbf{R},$

Different methods were needed for global existence/blow-up in

$$-\infty < a < -2, \quad -2 \leqq a < -1, \quad -1 \leqq a < 0, \quad 0 \leqq a < 1$$

- The case of -∞< a < -2 is settled in Zhu & O., Taiwanese J. Math. (2000).

$$\phi(t) \equiv \int_0^1 u_x(t,x)^2 dx$$

$$\frac{d^2}{dt^2}\phi(t) \geq b\phi(t)^3$$

$-2 \leqq a < -1$.   Follows the recipe of Hunter & Saxton (’91)

- Use the Lagrangian coordinates

$$X_t = u(t, X(t,\xi)), \quad X(0,\xi) = \xi, \quad (0 \le \xi \le 1)$$

- Define $V(t,\xi) = X_\xi(t,\xi)$.

$$VV_{tt} = (V_t)^2 - I(t)V, \quad I(t) = \int_0^1 \frac{V_t^2}{V}\, d\xi$$

- V tends to $-\infty$.
- Global weak solution in the case of a= -2 (Bressan & Constantin ‘05).

Blow-up occurs both in $-\infty < a < -2$ and in $-2 \leqq a < -1$, but

- Asymptotic behavior is quite different.
- $\|u_x(t)\|_{L^2}$ blow up. $(-\infty < a < -2)$
- $\|u_x(t)\|_{L^2}$ is bounded. $\|u_x(t)\|_{L^\infty}$ blows up.

  $(-2 \leqq a < -1)$

$-1 \leqq a < 0$. Follows the recipe of Chen &O. Proc. Japan Acad., (2002)

- Define $\Phi(s) = |s|^{-1/a}$
- Invariance

$$\frac{d}{dt}\int_0^1 \Phi(u_{xx}(t,x))dx = \int_0^1 \Phi'(u_{xx})[-uu_{xxx} + au_x u_{xx}]dx$$

$$= \int_0^1 [\Phi(u_{xx}) + au_{xx}\Phi'(u_{xx})]u_x dx = 0.$$

- Boundedness of $\int_0^1 \left|u_{xx}(t,x)\right|^{-1/a} dx, \quad \int_0^1 \left|u_{xx}(t,x)\right| dx$

-1 $\leqq a < 0$. Continued.

- $\|u_x(t)\|_\infty \le c$

- $u_{txx} + uu_{xxx} - au_x u_{xx} = \nu\, u_{xxxx}$ gives us

$$\frac{d}{dt}\int_0^1 u_{xx}(t,x)^2\,dx = (2a+1)\int_0^1 u_x u_{xx}{}^2\,dx$$

$$\frac{d}{dt}\int_0^1 u_{xx}(t,x)^2\,dx \le c(2a+1)\int_0^1 u_{xx}(t,x)^2\,dx$$

$0 \leqq a < 1.$ Follows the recipe of Chen &O. Proc. Japan Acad., (2002)

- Define $$\Phi(s) = \begin{cases} |s|^{1/(1-a)} & (s < 0) \\ 0 & (0 < s) \end{cases}$$

- Then $$\frac{d}{dt}\int_0^1 \Phi(u_{xxx})dx = a\int_0^1 u_{xx}^2 \Phi'(u_{xxx})dx \leq 0$$

- $\int_0^1 \left|u_{xxx}(t,x)\right|dx$ is bounded.

Non-smooth, self-similar blow-up solutions when $-1 < a < +\infty$

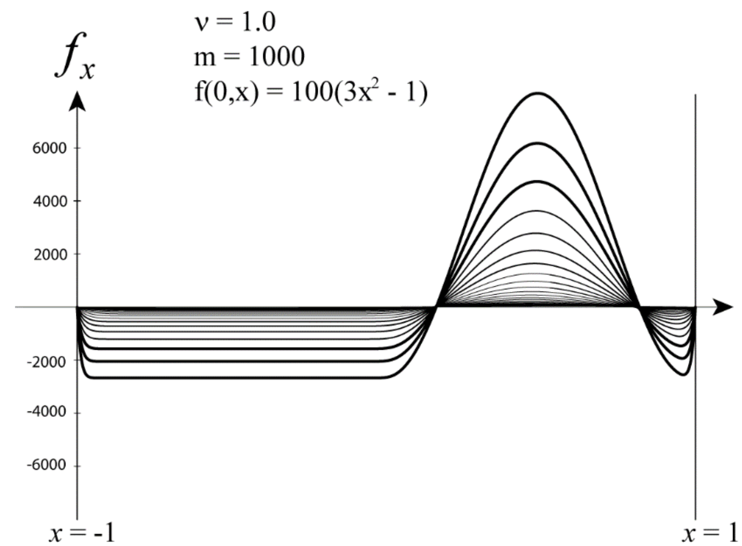
$$u(t,x) = \frac{F(x)}{T-t}$$

$$F'' + FF''' - aF'F'' = 0.$$

- **Nontrivial solution exists for all $-1 < a < +\infty$.**

Another

- 3D Navier-Stokes exact sol.



$$f_{txx} + (f - Sf) f_{xxx} - (f_x - (Sf)_x) f_{xx} = \nu f_{xxxx}$$

$$Sf(t,x) = f(t,-x)$$

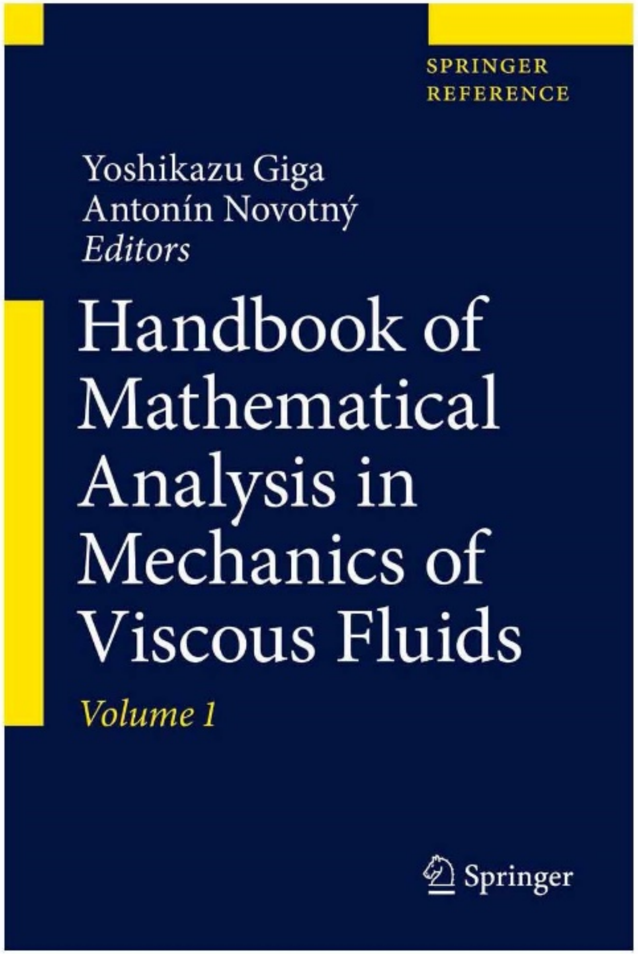
- Nagayama and O., '02 numerical experiment.
- Proof ???

Conclusions

- A proper convection term prevents the solution from blowing-up. Or, at least, rapid growth is slowed down by a convection term
- There are some cases where proof is needed.
- Blow-up behavior is very different from a nonlinear heat eqn: *the yoke of non-locality.*

- More problems in Bae, Chae & O. Nonlinear Analysis 2017. O. in Handbook

Thank you very much.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