Direct and inverse bifurcation problems and related topics II

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Outline

We consider:

$$\begin{aligned} -\Delta u + f(u) &= \lambda u \quad \text{in } \Omega, \\ u &> 0, \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

- $\Omega \subset \mathbf{R}^N$: appropriately smooth bounded domain.
- $\lambda > 0$: a parameter.

We assume that $f(u)$ is **unknown to** satisfy the conditions (A.1)–(A.3):

(**A.1**) $f(u)$ is a function of C^1 for $u \ge 0$ satisfying $f(0) = f'(0) = 0$.

(**A.2**) $f(u)/u$ is strictly increasing for $u \ge 0$.

(**A.3**) $f(u)/u \to \infty$ as $u \to \infty$.

Examples

Examples of $f(u)$ which satisfy (A.1)–(A.3)

$$
\begin{aligned}
f(u) &= u^p \quad (p > 1), \\
f(u) &= u^p + u^m \quad (p > m > 1).
\end{aligned}
$$

The First Purpose

We study inverse bifurcation problems of in L^q-framework $(1 \le q \le \infty)$.
In particular:

・From mathematical point of view, since (1.1) is regarded as an eigenvalue problem, it seems natural to treat it in L^2-framework.

・From biological point of view, if $f(u) = u^2$, then (1.1) is the model equation of population density of some species. Therefore, it seems also important to treat it in L^1-framework.

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L^q-Bifurcation Curve

(1) Let $1 \le q \le \infty$ be fixed. Let $\|\cdot\|_q$ be L^q-norm. For any given $\alpha > 0$, there exists a unique solution pair

$$(\lambda, u) = (\lambda(q,\alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{\Omega})$$

such that

$$\|u_\alpha\|_q = \alpha.$$

(2) The following set gives all the solutions of (1.1):

$$\{(\lambda(q,\alpha), u_\alpha) : \alpha > 0\} \subset \mathbf{R}_+ \times C^2(\bar{\Omega})$$

(3)

$$\lambda(q,\alpha) \to \lambda_1 \quad (\alpha \to 0,\ \ \lambda_1 : \text{ the first eigenvalue of } \ -\Delta_D),$$

$$\lambda(q,\alpha) \nearrow \infty \quad (\alpha \to \infty).$$

L^q-Bifurcation Curve

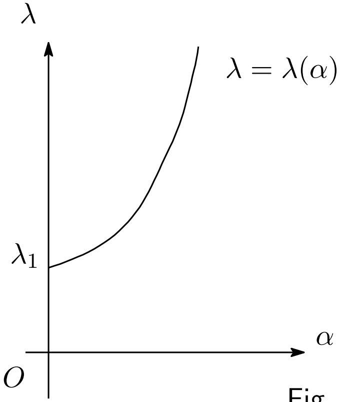


Fig. 1

L^2-framework

Let $f(u) = f_1(u)$ and $f(u) = f_2(u)$ be unknown to satisfy (A.1)–(A.3). Furthermore, let

$$F_j(u) := \int_0^u f_j(s)ds \qquad (j = 1, 2).$$

Assume that F_1 and F_2 satisfy the following condition (B.1).

(**B.1**) Let

$$W := \{u \geq 0 : F_1(u) = F_2(u)\}$$

Then W consists, at most, of the (finite or infinite numbers of) intervals and the points $\{u_n\}_{n=1}^{\infty}$ whose accumulation point is only ∞.

Theorem 1.1

Theorem 1.1.(S, 2009) *Assume that* f_1 *and* f_2 *are unknown to satisfy (A.1)–(A.3) and (B.1). Furthermore, if* $N \geq 2$*, then assume that* f_1 *and* f_2 *satisfy the following (A.4).*

(A.4) For $u, v \geq 0$*,*

$$F_j(u+v) \leq C(F_j(u) + F_j(v)) \qquad (j = 1, 2).$$

Suppose

$$\lambda_1(2, \alpha) = \lambda_2(2, \alpha) \quad for \;\; any \;\; \alpha > 0.$$

Here, $\lambda_j(2, \alpha)$ *is the* L^2*-bifurcation curve associated with* $f(u) = f_j(u)$ *(*$j = 1, 2$*). Then* $f_1(u) \equiv f_2(u)$ *for* $u \geq 0$*.*

The proof depends on the variational method.

Proof of Theorem 1.1

Variational Structure: Critical value $C_1(\alpha)$ and $C_2(\alpha)$.

For simplicity, let $\Omega = I = (0, 1)$. For $j = 1, 2$ and $v \in H_0^1(I)$, let

$$\Phi_j(v) := \frac{1}{2}\|v'\|_2^2 + \int_0^1 F_j(v(t))dt. \tag{1.2}$$

For $\alpha > 0$, we put

$$M_\alpha := \{v \in H_0^1(I) : \|v\|_2 = \alpha\}.$$

Proof of Theorem 1.1

For $j = 1, 2$ and $\alpha > 0$ we put

$$C_j(\alpha) := \min\{\Phi_j(v) : v \in M_\alpha\}. \tag{1.3}$$

Existence of unique positive minimizer

By taking a minimizing sequence, Lagrange multiplier theorem and strong maximum principle, there exists a Lagrange multiplier $\lambda_j(\alpha)$ and a unique minimizer $u_{j,\alpha} \in M_\alpha$ which satisfies (1.1) with $f = f_j$.

Proof of Theorem 1.1

The relationship between $C_j(\alpha)$ and $\lambda_j(\alpha)$

By direct calculation, we obtain

$$\frac{dC_j(\alpha)}{d\alpha} = 2\lambda_j(\alpha)\alpha.$$

By this, we obtain

Lemma 1.2. $C_1(\alpha) = C_2(\alpha)$ *for* $\alpha \geq 0$.

Proof. Since $C_1(0) = C_2(0) = 0$, we obtain,

$$\begin{aligned} C_1(\alpha) &= \int_0^\alpha \frac{d}{ds} C_1(s) ds = \int_0^\alpha 2\lambda_1(s) s ds \\ &= \int_0^\alpha 2\lambda_2(s) s ds = C_2(\alpha). \end{aligned}$$

Proof of Theorem 1.1

Clearly, $0 \in W$, where

$$W := \{u \geq 0 : F_1(u) = F_2(u)\}.$$

(**a**) Assume that $0 \in W$ is contained in the interval $[0, \epsilon]$ for some constant $0 < \epsilon \ll 1$. This implies that for $0 \leq u \leq \epsilon$,

$$F_1(u) = F_2(u).$$

Proof of Theorem 1.1

Let K be a connected component of W satisfying $[0, \epsilon] \subset K$. Then $K = [0, u_1]$. If $u_1 < \infty$, then without loss of generality, by (B.1), there exists a constant $0 < \epsilon \ll 1$ such that

$$\begin{aligned} F_1(u) &= F_2(u) \quad (0 \le u \le u_1), \\ F_1(u) &< F_2(u), \quad (u_1 < u < u_1 + \epsilon). \end{aligned}$$

Now we choose $\alpha > 0$ satisfying

$$\|u_{2,\alpha}\|_\infty = u_1 + \epsilon.$$

Proof of Theorem 1.1

Then

$$\begin{aligned} C_1(\alpha) &\le \Phi_1(u_{2,\alpha}) = \frac{1}{2}\|u'_{2,\alpha}\|_2^2 + \int_0^1 F_1(u_{2,\alpha}(t))dt \\ &< \frac{1}{2}\|u'_{2,\alpha}\|_2^2 + \int_0^1 F_2(u_{2,\alpha}(t))dt \\ &= \Phi_2(u_{2,\alpha}) = C_2(\alpha). \end{aligned}$$

This contradicts Lemma 1.2. Therefore, we see that $u_1 = \infty$ and $K = [0, \infty)$. This implies $F_1(u) \equiv F_2(u)$, and consequently, $f_1(u) \equiv f_2(u)$.

Proof of Theorem 1.1

(**b**) Assume that $0 \in W$ is an isolated point in W. Then by (B.4), without loss of generality, there exists a constant $0 < \epsilon \ll 1$ such that

$$F_1(u) < F_2(u)$$

for $0 < u < \epsilon$. Then by the same argument as that in (a) just above, we can derive a contradiction. Therefore, the case (b) does not occur.
From (a) and (b), we obtain our conclusion. □

We consider the following nonlinear eigenvalue problems

$$-u''(t) = \lambda\left(u(t) + g(u(t))\right), \quad t \in I =: (-1,1), \tag{2.1}$$

$$u(t) > 0, \quad t \in I, \tag{2.2}$$

$$u(-1) = u(1) = 0, \tag{2.3}$$

where $g(u) \in C(\bar{\mathbb{R}}_+)$ and $\lambda > 0$ is a parameter.

oscillating bifurcation curve

It is well known (cf. [T. Laetsch, 1970]) that, if, for example,

$$u + g(u) > 0 \qquad \text{for} \quad u > 0,$$

then by **time-map method**, we find that λ is parameterized by using $\alpha = \|u\|_\infty$, such as $\lambda = \lambda(\alpha)$ and is a continuous function of $\alpha > 0$. Since λ depends on g, we write

$$\lambda = \lambda(g, \alpha).$$

One of the nonlinear terms $g(u)$ we are interested in is

$$g_1(u) = \sin \sqrt{u}.$$

In this case, the equation (2.1)–(2.3) has been proposed in Cheng (2002) as a model problem which has arbitrary many solutions near $\lambda = \pi^2/4$.

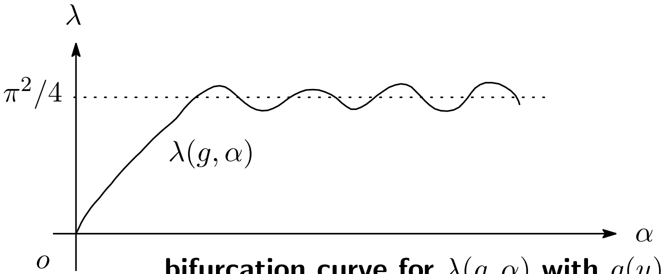
Theorem 2.0 ([Cheng, 2002]). *Let* $g(u) = \sin\sqrt{u}$ *(*$u \geq 0$*). Then for any integer* $r \geq 1$*, there is* $\delta > 0$ *such that if* $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$*, then (1.1)–(1.3) has at least* r *distinct solutions.*

oscillating bifurcation curve

- Certainly, Theorem 2.0 gives us the imformation about the solution set of (2.1)–(2.3), and we expect that $\lambda(\alpha)$ oscillates and intersects the line $\lambda = \pi^2/4$ infinitely many times as $\alpha \to \infty$.

- So we expect that the bifurcation curve for g_1 is as follows.

Structure of the bifurcation curve for $g(u) = \sin\sqrt{u}$



bifurcation curve for $\lambda(g, \alpha)$ **with** $g(u) = \sin\sqrt{u}$

Structure of the bifurcation curve for $g(u) = \sin\sqrt{u}$

- The first purpose here is to prove the expectation above is valid.

- Precisely, we establish the asymptotic formula for $\lambda(g, \alpha)$ as $\alpha \to \infty$, which gives us the well understanding why $\lambda(g, \alpha)$ intersect the line $\lambda = \pi^2/4$ infinitely many times.

- We also obtain the asymptotic formula for $\lambda(g, \alpha)$ as $\alpha \to 0$. These two formulas clarify the total structure of $\lambda(g, \alpha)$.

Asymptotic length of bifurcation curve

We also consider the asymptotic length of $\lambda(g, \alpha)$ $(\alpha \gg 1)$ defined by

$$L(g, \alpha) := \int_{\alpha}^{2\alpha} \sqrt{1 + (\lambda'(g, s))^2} ds. \tag{2.4}$$

In particular, we are interested in $g(u)$, which satisfies

$$L(g, \alpha) = \alpha + o(\alpha), \quad (\alpha \to \infty). \tag{2.5}$$

This notion will be used to propose a new concept of inverse bifurcation problem.

Global behavior of bifurcation curve for $g(u) = \sin\sqrt{u}$

Theorem 2.1. *Let* $g(u) = g_1(u) = \sin\sqrt{u}$. *Then as* $\alpha \to \infty$,

$$\lambda(g_1, \alpha) = \frac{\pi^2}{4} - \pi^{3/2}\alpha^{-5/4}\cos\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + o(\alpha^{-5/4}), \quad (2.6)$$

$$\lambda'(g_1, \alpha) = \frac{1}{2}\pi^{3/2}\alpha^{-7/4}\sin\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + o(\alpha^{-7/4}), \quad (2.7)$$

$$L(g_1, \alpha) = \alpha + \frac{1}{40}\left(1 - \frac{1}{4\sqrt{2}}\right)\alpha^{-5/2} + o(\alpha^{-5/2}). \quad (2.8)$$

Local behavior of bifurcation curve for $g(u) = \sin\sqrt{u}$

Theorem 2.2. *Let* $g(u) = g_1(u) = \sin\sqrt{u}$.

(i) As $\alpha \to 0$*, the following asymptotic formula for* $\lambda(g_1, \alpha)$ *holds:*

$$\lambda(g_1, \alpha) = \frac{3}{4}C_1^2\sqrt{\alpha} + \frac{3}{2}C_1C_2\alpha + O(\alpha^{3/2}), \tag{2.9}$$

where

$$C_1 := \int_0^1 \frac{1}{\sqrt{1 - s^{3/2}}}ds, \quad C_2 := -\frac{3}{8}\int_0^1 \frac{1 - s^2}{\sqrt{1 - s^{3/2}}}ds. \tag{2.10}$$

Local behavior of bifurcation curve for $g(u) = \sin\sqrt{u}$

(ii) *Let v_0 be a unique classical solution of the following equation*

$$-v_0''(t) = \frac{3}{4}C_1^2\sqrt{v_0(t)}, \quad t \in I, \tag{2.11}$$

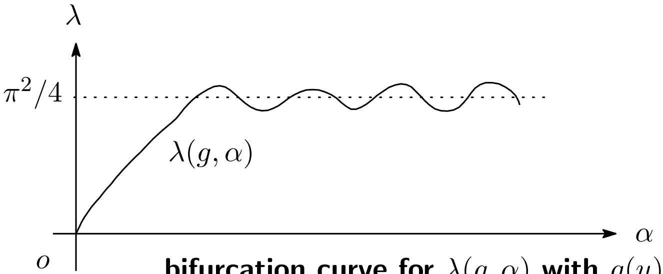
$$v_0(t) > 0, \quad t \in I, \tag{2.12}$$

$$v_0(-1) = v_0(1) = 0. \tag{2.13}$$

Furthermore, let $v_\alpha(t) := u_\alpha(t)/\alpha$. Then $v_\alpha \to v_0$ in $C^2(I)$ as $\alpha \to 0$.

・For the uniqueness of the positice solution of (2.11)–(2.13), we refer to A. Ambrosetti, H. Brezis, G. Cerami (1994).

Structure of the bifurcation curve for $g(u) = \sin\sqrt{u}$



bifurcation curve for $\lambda(g, \alpha)$ with $g(u) = \sin\sqrt{u}$

Oscillating bifurcation curve

The other nonlinear terms we treat in this talk are

$$g_2(u) = \frac{1}{2}\sin u, \tag{2.14}$$

$$g_3(u) = \sin u^2. \tag{2.15}$$

We know that the shape of $\lambda(g_2, \alpha)$ is something like Fig.2 below.

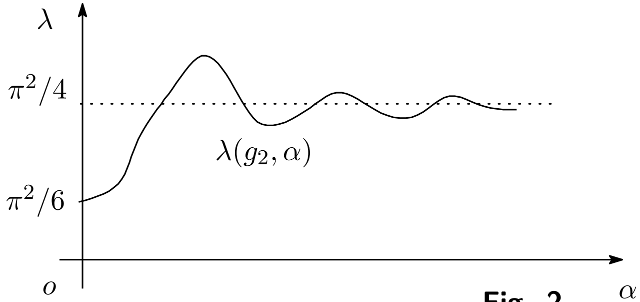


Fig. 2

Theorem 2.3. *Let* $g(u) = g_2(u) = (1/2)\sin u$*. Then as* $\alpha \to \infty$

$$\lambda(g_2, \alpha) = \frac{\pi^2}{4} - \frac{\pi}{2\alpha}\sqrt{\frac{\pi}{2\alpha}}\sin\left(\alpha - \frac{1}{4}\pi\right) + O(\alpha^{-2}), \tag{2.16}$$

$$\lambda'(g_2, \alpha) = -\frac{\pi}{2\alpha}\sqrt{\frac{\pi}{2\alpha}}\cos\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-3/2}), \tag{2.17}$$

$$L(g_2, \alpha) = \alpha + \frac{3\pi^3}{256}\alpha^{-2} + o(\alpha^{-2}). \tag{2.18}$$

Global structure of the bifurcation curve for $g(u) = \sin u^2$

Theorem 2.4. *Let* $g(u) = g_3(u) = \sin u^2$. *Then as* $\alpha \to \infty$,

$$\lambda(g_3, \alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{2}\alpha^{-2}\cos\left(\alpha^2 - \frac{3}{4}\pi\right) + o(\alpha^{-2}), \quad (2.19)$$

$$\lambda'(g_3, \alpha) = \frac{\pi^{3/2}}{\alpha}\sin\left(\alpha^2 - \frac{3}{4}\pi\right) + o(\alpha^{-1}). \quad (2.20)$$

$$L(g_3, \alpha) = \alpha + \frac{\pi^3}{8\alpha} + o(\alpha^{-1}). \quad (2.21)$$

Local behavior of the bifurcation curve for $g(u) = \sin u^2$

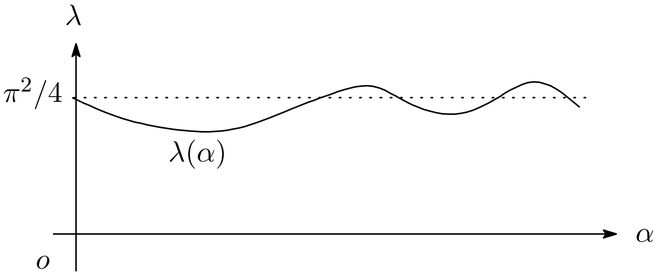
Theorem 2.5. *Let* $g(u) = g_3(u) = \sin u^2$*. Then as* $\alpha \to 0$*,*

$$\lambda(g_3, \alpha) \;=\; \frac{\pi^2}{4} - \frac{1}{3}\pi A_1 \alpha + \left(\frac{1}{9}A_1^2 + \frac{1}{6}\pi A_2\right)\alpha^2 + o(\alpha^2), \quad (2.22)$$

where

$$A_1 = \int_0^1 \frac{1 - s^3}{(1 - s^2)^{3/2}} ds, \quad A_2 = \int_0^1 \frac{(1 - s^3)^2}{(1 - s^2)^{5/2}} ds. \quad (2.23)$$

Structure of the bifurcation curve for $g(u) = \sin u^2$



bifurcation curve for $\lambda(\alpha)$ **with** $g(u) = \sin u^2$

Inverse problem A

Assume that

$$g \in \Lambda := \{g \in C(\bar{\mathbb{R}}_+) : \lambda(g, \alpha) \to \pi^2/4 \text{ as } \alpha \to \infty\}$$

satisfies

$$L(g, \alpha) = \alpha + o(\alpha), \quad (\alpha \to \infty). \tag{2.24}$$

Then is it possible to distinguish g from g_i ($i = 1, 2, 3$) by the second term of $L(g, \alpha)$?

Inverse Problem A (Weak Version)

・This approach for inverse bifurcation problem seems to be **new**, and it is significant to consider whether this framework is suitable or not, since a few attempts have so far been made.

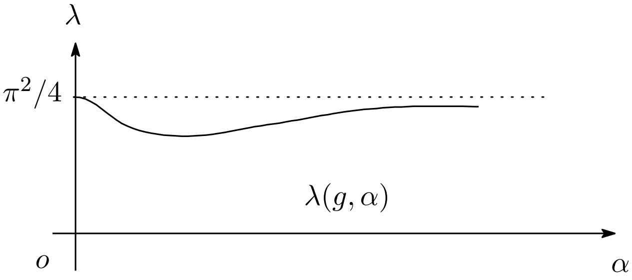
・We restrict our attention to the **'monotone' nonlinear terms** and make the simple approach to Inverse problem A.

Inverse Problem A（Weak Version)

Assume that $g(u) \in C^1(\bar{\mathbb{R}}_+)$ satisfies the following assumption (C.1).

(C.1) $g(0) = g'(0) = 0$, $g'(u) \geq 0$ for $u > 0$ and $g(u) = Cu^m$ for $u \geq 1$, where $C > 0$ and $0 < m < 1$ are constants.

Graph of $\lambda(g, \alpha)$ ($g(u)$ is "monotome" type)



bifurcation curve for $g(u) \sim Cu^m$

Answer to Inverse Problem A

Theorem 2.6. *Let $g(u)$ satisfy (**C.1**). Then as $\alpha \to \infty$,*

$$L(g,\alpha) = \alpha + \frac{2^{2m-3}-1}{2(2m-3)} A(m)^2 \alpha^{2m-3} + o(\alpha^{2m-3}), \qquad (2.25)$$

$$\lambda(g,\alpha) = \frac{\pi^2}{4} - \frac{\pi}{m+1} CC(m)\alpha^{m-1} + o(\alpha^{m-1}), \qquad (2.26)$$

$$\lambda'(g,\alpha) = -\frac{m-1}{m+1} \pi CC(m)\alpha^{m-2} + o(\alpha^{m-2}), \qquad (2.27)$$

where

$$A(m) := \frac{(1-m)\pi CC(m)}{1+m}, \quad C(m) = \int_0^1 \frac{1-s^{m+1}}{(1-s^2)^{3/2}} ds. \qquad (2.28)$$

Answer to Inverse Problem A (Weak Version)

$$g_1(u) = \sin\sqrt{u}, \quad g_2(u) = \frac{1}{2}\sin u, \quad g_3(u) = \sin u^2,$$

and $g(u)$ is a "monotone type" $(0 < m < 1)$. Then

$$\begin{aligned} L(g_1, \alpha) &= \alpha + \frac{1}{40}\left(1 - \frac{1}{4\sqrt{2}}\right)\alpha^{-5/2} + o(\alpha^{-5/2}), \\ L(g_2, \alpha) &= \alpha + \frac{3\pi^3}{256}\alpha^{-2} + o(\alpha^{-2}), \\ L(g_3, \alpha) &= \alpha + \frac{\pi^3}{8}\alpha^{-1} + o(\alpha^{-1}), \\ L(g, \alpha) &= \alpha + \frac{2^{2m-3} - 1}{2(2m-3)}A(m)^2\alpha^{2m-3} + o(\alpha^{2m-3}). \end{aligned}$$

- **We can distinguish g and g_3 by the second term of L, but if we put $m = 1/4$ in $L(g_1, \alpha)$, $m = 1/2$ in $L(g_2, \alpha)$, and choose C suitably, we can not distinguish g and g_1, g_2 by the second term of L.**

Proof of Theorems

$=$ **Time map**

$+$ **Asymptotic formulas for some special functions.**

・The proofs of the Theorems in this section basically depend on the time-map argument. In particular, the key tool of the proof of Theorem 2.1 is the asymptotic formula for the Bessel functions obtained by Krasikov (2016).

The case $g(u) = \sin u^2$ and $\alpha \gg 1$

In this section, let $g(u) = g_3(u) = \sin u^2$ and $\alpha \gg 1$. For simplicity, we write $\lambda = \lambda(\alpha)$. For $u \geq 0$, let

$$G(u) := \int_0^u g(s)ds = \int_0^u \sin t^2 dt = \sqrt{\frac{\pi}{2}} S(u), \tag{3.1}$$

where $S(u)$ is the Fresnel sine integral defined by

$$S(u) = \sqrt{\frac{2}{\pi}} \int_0^u \sin x^2 dx. \tag{3.2}$$

Further, let $C(\alpha)$ be the Fresnel cosine integral defined by

$$C(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\alpha \cos x^2 dx. \tag{3.3}$$

The case $g_3(u) = \sin u^2$ and $\alpha \gg 1$

Then we know (cf. [I. S. Gradshteyn and I. M. Ryzhik (2015), pp. 898-899]) that as $\alpha \to \infty$,

$$S(\alpha) = \frac{1}{2} - \frac{1}{\sqrt{2\pi\alpha}} \cos^2 \alpha + O(\alpha^{-2}), \tag{3.4}$$

$$C(\alpha) = \frac{1}{2} + \frac{1}{\sqrt{2\pi\alpha}} \sin^2 \alpha + O(\alpha^{-2}). \tag{3.5}$$

The case $g_3(u) = \sin u^2$ and $\alpha \gg 1$

It is known that if $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (2.1)–(2.3), then

$$u_\alpha(t) = u_\alpha(-t), \quad 0 \le t \le 1, \tag{3.6}$$

$$u_\alpha(0) = \max_{-1 \le t \le 1} u_\alpha(t) = \alpha, \tag{3.7}$$

$$u_\alpha'(t) > 0, \quad -1 < t < 0. \tag{3.8}$$

Time map $\lambda(g_3, \alpha)$: $(g_3(u) = \sin u^2)$

By (2.1), we have

$$\left\{u_\alpha''(t) + \lambda\left(u_\alpha(t) + \sin\sqrt{u_\alpha(t)}\right)\right\} u_\alpha'(t) = 0.$$

By this, we obtain

$$\frac{1}{2}u_\alpha'(t)^2 + \lambda\left(\frac{1}{2}u_\alpha(t)^2 + G(u_\alpha(t))\right) = \text{constant} = \lambda\left(\frac{1}{2}\alpha^2 + G(\alpha)\right).$$

This along with (3.8) implies that for $-1 \le t \le 0$,

$$u_\alpha'(t) = \sqrt{\lambda}\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}. \tag{3.9}$$

For $0 \le s \le 1$, we have

$$\left|\frac{G(\alpha) - G(\alpha s)}{\alpha^2(1 - s^2)}\right| = \left|\frac{\int_{\alpha s}^{\alpha} g(t)dt}{\alpha^2(1 - s^2)}\right| \le \frac{\alpha(1 - s)}{\alpha^2(1 - s^2)} \le \alpha^{-1}. \tag{3.10}$$

Time map $\lambda(g_3, \alpha)$: $(g_3(u) = \sin u^2)$

By (3.9), (3.10), putting $s := u_\alpha(t)/\alpha$ and Taylor expansion, we obtain

$$\begin{aligned}
\sqrt{\lambda} &= \int_{-1}^{0} \frac{u'_\alpha(t)}{\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}} dt \qquad (3.11)\\
&= \int_0^1 \frac{1}{\sqrt{1 - s^2 + 2(G(\alpha) - G(\alpha s))/\alpha^2}} ds \\
&= \int_0^1 \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1 + 2(G(\alpha) - G(\alpha s))/(\alpha^2(1-s^2))}} ds \\
&= \int_0^1 \frac{1}{\sqrt{1-s^2}} \left\{ 1 - \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)}(1 + o(1)) \right\} ds \\
&= \frac{\pi}{2} - \frac{1}{\alpha^2}(1 + o(1)) \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1-s^2)^{3/2}} ds.
\end{aligned}$$

Time map $\lambda(g_3, \alpha)$: $(g_3(u) = \sin u^2)$

We put

$$K := \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1 - s^2)^{3/2}} ds. \tag{3.12}$$

Lemma 3.1. *As $\alpha \to \infty$,*

$$K = \frac{\sqrt{\pi}}{2}(1 + o(1)) \cos\left(\alpha^2 - \frac{3}{4}\pi\right). \tag{3.13}$$

Proof. For $0 \le \theta \le \pi/2$, we put

$$M(\theta) := G(\alpha) - G(\alpha s) = \int_{\alpha \sin\theta}^{\alpha} \sin t^2 dt. \tag{3.14}$$

We put $s = \sin\theta$ in (3.12). Then by (3.1), (3.14) and integration by parts, we obtain

Time map $\lambda(g_3, \alpha)$: $(g_3(u) = \sin u^2)$

$$
\begin{aligned}
K &= \int_0^{\pi/2} \frac{1}{\cos^2\theta} M(\theta) d\theta \qquad (3.15)\\
&= [\tan\theta M(\theta)]_0^{\pi/2} + \alpha \int_0^{\pi/2} \tan\theta \sin(\alpha\sin\theta)^2 \cos\theta d\theta \\
&:= K_1 + \alpha K_2.
\end{aligned}
$$

Since

$$
\lim_{\theta\to\pi/2} \frac{M(\theta)}{\cos\theta} = \lim_{\theta\to\pi/2} \frac{\alpha\cos\theta\sin(\alpha\sin\theta)^2}{\sin\theta} = 0, \qquad (3.16)
$$

we see that $K_1 = 0$. Now we calculate K_2.

$$\begin{aligned} K_2 &= \int_0^{\pi/2} \sin\theta \sin(\alpha \sin\theta)^2 d\theta \\ &= \int_0^{\pi/2} \sin\theta \sin(\alpha^2 - \alpha^2 \cos^2\theta) d\theta \\ &= \sin\alpha^2 \int_0^{\pi/2} \sin\theta \cos(\alpha^2 \cos^2\theta) d\theta \\ &\quad - \cos\alpha^2 \int_0^{\pi/2} \sin\theta \sin(\alpha^2 \cos^2\theta) d\theta \\ &= K_{21} \sin\alpha^2 - K_{22} \cos\alpha^2. \end{aligned} \tag{3.17}$$

By putting $t = \cos\theta$, we obtain by (3.5) that as $\alpha \to \infty$,

$$K_{21} = \int_0^1 \cos(\alpha^2 t^2) dt = \frac{1}{\alpha}\int_0^\alpha \cos x^2 dx = \sqrt{\frac{\pi}{2}}\frac{1}{2\alpha}(1 + o(1)). \tag{3.18}$$

By the same calculation as that to obtain (3.18), we obtain

$$K_{22} = \int_0^1 \sin(\alpha^2 t^2) dt = \frac{1}{\alpha}\sqrt{\frac{\pi}{2}} S(\alpha) = \sqrt{\frac{\pi}{2}}\frac{1}{2\alpha}(1 + o(1)). \tag{3.19}$$

By (3.17)–(3.19), we obtain

Time map $\lambda(g_3, \alpha)$: $(g_3(u) = \sin u^2)$

$$K = \frac{1}{2}\sqrt{\frac{\pi}{2}}(1+o(1))(\sin\alpha^2 - \cos\alpha^2) \tag{3.20}$$
$$= \frac{\sqrt{\pi}}{2}(1+o(1))\sin\left(\alpha^2 - \frac{1}{4}\pi\right)$$
$$= \frac{\sqrt{\pi}}{2}(1+o(1))\cos\left(\alpha^2 - \frac{3}{4}\pi\right).$$

This implies (3.13). Thus the proof is complete. □

By Lemma 3.1 and (3.11), we obtain $\lambda(g_3, \alpha)$. □

How to obtain $L(g_3, \alpha)$.

$$
\begin{aligned}
L(g_3,\alpha) &= \int_\alpha^{2\alpha} \sqrt{1 + \frac{\pi^3}{t^2}(1+o(1))\sin^2\left(t^2 - \frac{3\pi}{4}\right)} dt \qquad (3.21) \\
&= \int_\alpha^{2\alpha} 1 + \frac{\pi^3}{2t^2}(1+o(1))\sin^2\left(t^2 - \frac{3\pi}{4}\right) dt \\
&= \alpha + \frac{\pi^3}{4}(1+o(1)) \int_\alpha^{2\alpha} \left(\frac{\sin^2 t^2}{t^2} + \frac{\cos^2 t^2}{t^2} + \frac{2\sin t^2 \cos t^2}{t^2}\right) dt.
\end{aligned}
$$

Clearly,

$$
\int_\alpha^{2\alpha} \left(\frac{\sin^2 t^2}{t^2} + \frac{\cos^2 t^2}{t^2}\right) dt = \int_\alpha^{2\alpha} \frac{1}{t^2} dt = \frac{1}{2\alpha}. \qquad (3.22)
$$

Furthermore, Then by integration by parts, we obtain

Asymptotic length of bifurcation curve

$$
\begin{aligned}
\int_\alpha^{2\alpha} \frac{2\sin t^2 \cos t^2}{t^2} dt &= \int_\alpha^{2\alpha} \frac{\sin(2t^2)}{t^2} dt \qquad (3.23) \\
&= \sqrt{2} \int_{\sqrt{2}\alpha}^{2\sqrt{2}\alpha} \frac{\sin x^2}{x^2} \\
&= \sqrt{2} \left[-\frac{1}{2t^3} \cos t^2 \right]_{\sqrt{2}\alpha}^{2\sqrt{2}\alpha} + \frac{3\sqrt{2}}{2} \int_{\sqrt{2}\alpha}^{2\sqrt{2}\alpha} \frac{\cos t^2}{t^4} dt \\
&= -\frac{1}{32\alpha^3} \cos(8\alpha^2) + \frac{1}{4\alpha^3} \cos 2\alpha^2 + O(\alpha^{-3}).
\end{aligned}
$$

Thus the proof is complete. □

The case $g(u) = u^p \sin(u^q)$ $(0 \le p < 1, 0 < q \le 1)$

Now we consider the precise asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to \infty$ for

$$g(u) = u^p \sin(u^q) \quad (0 \le p < 1, 0 < q \le 1).$$

We prove the following Theorem 3.1 by the time-map argument and the stationary phase method. We have to be careful about the regularity of the functions which will be appear after the time-map argument.

Theorem 4.1.([S, 22]) *Let* $g(u) = u^p \sin(u^q)$, *where* $0 \le p < 1$ *and* $0 < q \le 1$ *are fixed constants. Then as* $\alpha \to \infty$,

$$\lambda(\alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{\sqrt{2q}} \alpha^{p-1-(q/2)} \sin\left(\alpha^q - \frac{\pi}{4}\right) + o(\alpha^{p-1-(q/2)}). \quad (4.1)$$

Local behavior of $\lambda(\alpha)$

Next, to understand the whole structure of $\lambda(\alpha)$ in detail, we establish the asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to 0$.

Theorem 4.2.([S, 22]) *Let* $g(u) = u^p \sin(u^q)$, *where* $0 \le p < 1$, $0 < q \le 1$. *Then the following asymptotic formulas hold* $\alpha \to 0$.

(i) Assume that $p + q > 1$. *Then*

$$\lambda(\alpha) = \frac{\pi^2}{4} - A_1 \pi \alpha^{p+q-1} + (A_1^2 + A_2 \pi)\alpha^{2(p+q-1)} + o(\alpha^{2(p+q-1)}), \tag{4.2}$$

where

$$A_1 = \frac{1}{p+q+1} \int_0^1 \frac{1 - s^{p+q+1}}{(1-s^2)^{3/2}} ds, \tag{4.3}$$

$$A_2 = \frac{3}{2(p+q+1)^2} \int_0^1 \frac{(1 - s^{p+q+1})^2}{(1-s^2)^{5/2}} ds. \tag{4.4}$$

Local behavior of $\lambda(\alpha)$

(ii) Assume that $p + q = 1$. *Then*

$$\lambda(\alpha) = \frac{\pi^2}{8} + \frac{\pi}{48} B\alpha^{2q} + o(\alpha^{2q}), \tag{4.5}$$

where

$$B = \frac{1}{q+1} \int_0^1 \frac{1 - s^{2q+2}}{(1 - s^2)^{3/2}} ds. \tag{4.6}$$

(iii) Assume that $p + q < 1 < p + 3q$. *Then*

$$\begin{aligned} \lambda(\alpha) &= \frac{p+q+1}{2} \alpha^{1-p-q} \\ &\quad \times \left\{ C_1^2 - \frac{p+q+1}{2} C_1 C_2 \alpha^{1-p-q} + o(\alpha^{1-p-q}) \right\}, \end{aligned} \tag{4.7}$$

where

$$C_1 = \int_0^1 \frac{1}{\sqrt{1 - s^{p+q+1}}} ds, \quad C_2 = \int_0^1 \frac{1 - s^2}{(1 - s^{p+q+1})^{3/2}} ds. \tag{4.8}$$

Local behavior of $\lambda(\alpha)$

(iv) Assume that $p+q<p+3q<1$. *Then*

$$\lambda(\alpha) = \frac{p+q+1}{2}\alpha^{1-p-q} \times \left\{ C_1^2 + \frac{p+q+1}{6(p+3q+1)} C_1 C_3 \alpha^{2q} + o(\alpha^{2q}) \right\}, \tag{4.9}$$

$$C_3 = \int_0^1 \frac{1-s^{p+3q+1}}{(1-s^{p+q+1})^{3/2}} ds. \tag{4.10}$$

(v) Assume that $p+q<p+3q=1$. *Then*

$$\lambda(\alpha) = \frac{p+q+1}{2}\alpha^{2q} \times \left\{ C_1^2 - \frac{5(p+q+1)}{12} C_1 C_4 \alpha^{2q} + o(\alpha^{2q}) \right\}, \tag{4.11}$$

$$C_4 = \int_0^1 \frac{1-s^2}{(1-s^{p+q+1})^{3/2}} ds. \tag{4.12}$$

Local behavior of $\lambda(\alpha)$

By Theorems 4.1 and 4.2, we understand that there exist three types of the asymptotic shapes of $\lambda(\alpha)$ (see figures below).

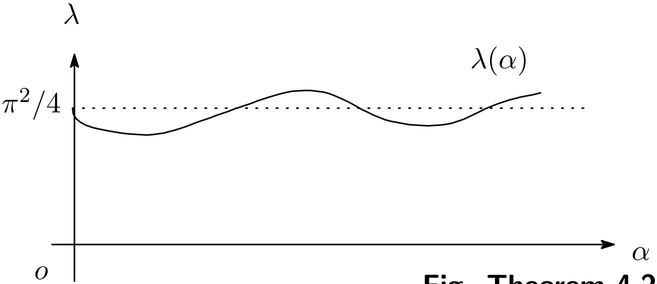


Fig. Theorem 4.2 (i)

Graph of $\lambda(\alpha)$

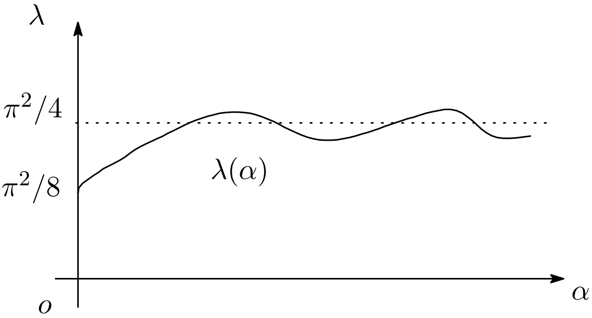


Fig. Theorem 4.2 (ii)

Graph of $\lambda(\alpha)$

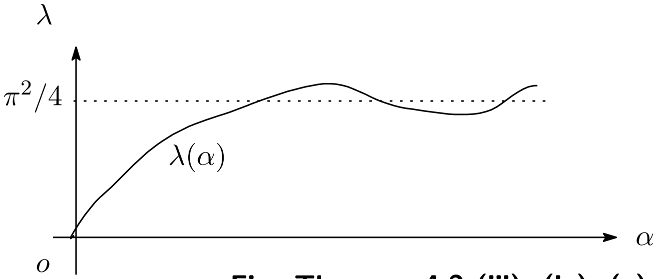


Fig. Theorem 4.2 (iii), (iv), (v)

Proofs: $g(u) = u^p \sin(u^q)$ $(0 \le p < 1, 0 < q \le 1)$

In this section, let $\alpha \gg 1$. Furthermore, we denote by C the various positive constants independent of α. For $u \ge 0$, let

$$g(u) = u^p \sin(u^q)$$

and

$$G(u) := \int_0^u g(s)ds. \tag{5.1}$$

Then by the same argument of time-map as that in Section 2, we obtain

Time-Map

$$\begin{aligned}
\sqrt{\lambda} &= \int_{-1}^{0} \frac{u_\alpha'(t)}{\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}} dt \qquad (5.2)\\
&= \int_0^1 \frac{1}{\sqrt{1 - s^2 + 2(G(\alpha) - G(\alpha s))/\alpha^2}} ds \\
&= \int_0^1 \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1 + 2(G(\alpha) - G(\alpha s))/(\alpha^2(1-s^2))}} ds \\
&= \int_0^1 \frac{1}{\sqrt{1-s^2}} \left\{ 1 - \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)} (1 + o(1)) \right\} ds \\
&= \frac{\pi}{2} - \frac{1}{\alpha^2}(1 + o(1)) \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1-s^2)^{3/2}} ds \\
&= \frac{\pi}{2} - \frac{1}{\alpha^2} K(\alpha)(1 + o(1)),
\end{aligned}$$

where

Key Lemma

$$K(\alpha) := \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1 - s^2)^{3/2}} ds. \tag{5.3}$$

To calculate $K(\alpha)$, we use the following Lemma. By combining [8, Lemma 2] and [10, Lemmas 2.25], we have following equalities.

Lemma 5.1. *Assume that the function* $f(r) \in C^2[0,1]$*, and* $h(r) = \cos(\pi r/2)$*. Then as* $\mu \to \infty$

$$\int_0^1 f(r) e^{i\mu h(r)} dr = e^{i(\mu - (\pi/4))} \sqrt{\frac{2}{\pi\mu}} f(0) + O\left(\frac{1}{\mu}\right). \tag{5.4}$$

In particular, by taking the imaginary part of (4.4),

$$\int_0^1 f(r) \sin(\mu h(r)) dr = \sqrt{\frac{2}{\pi\mu}} f(0) \sin\left(\mu - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu}\right). \tag{5.5}$$

Lemma 5.2. *As* $\alpha \to \infty$,

$$K(\alpha) = \sqrt{\frac{\pi}{2q}}\alpha^{p+1-(q/2)} \sin\left(\alpha^q - \frac{\pi}{4}\right) + o(\alpha^{p-1-(q/2)}). \tag{5.6}$$

PROOF. We put $s = \sin\theta$ in (5.3). Then by integration by parts,

$$\begin{aligned} K(\alpha) &= \int_0^{\pi/2} \frac{1}{\cos^2\theta}(G(\alpha) - G(\alpha\sin\theta))d\theta \qquad (5.7) \\ &= \int_0^{\pi/2} (\tan\theta)'(G(\alpha) - G(\alpha\sin\theta))d\theta \\ &= [\tan\theta(\, G(\alpha) - G(\alpha\sin\theta))]_0^{\pi/2} \\ &\quad + \alpha\int_0^{\pi/2} \tan\theta(\cos\theta(\alpha\sin\theta)^p \sin((\alpha\sin\theta)^q))d\theta. \end{aligned}$$

By l'Hôpital's rule, we obtain

$$\lim_{\theta\to\pi/2} \frac{\int_{\alpha\sin\theta}^{\alpha} y^p \sin(y^q)dy}{\cos\theta} = \lim_{\theta\to\pi/2} \frac{\alpha\cos\theta(\alpha\sin\theta)^p \sin((\alpha\sin\theta)^q)}{\sin\theta} = 0.$$

We put $m = 1/q$, $\sin^q \theta = \sin x$, $x = (\pi/2) - y$ and $y = (\pi/2)r$. Then

$$
\begin{aligned}
K(\alpha) &= \alpha^{p+1} \int_0^{\pi/2} \sin^{p+1}\theta \sin(\alpha^q \sin^q \theta) d\theta \qquad (5.8) \\
&= \frac{1}{q}\alpha^{p+1} \int_0^{\pi/2} \sin^{(p+2-q)/q} x \frac{\cos x}{\sqrt{1-\sin^{2m} x}} \sin(\alpha^q \sin x) dx \\
&= \frac{1}{q}\alpha^{p+1} \int_0^{\pi/2} \sin^{(p+2-q)/q} x \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin^{2m} x}} \sin(\alpha^q \sin x) dx \\
&= \frac{1}{q}\alpha^{p+1} \int_0^{\pi/2} \cos^{(p+2-q)/q} y \frac{\sqrt{1-\cos^2 y}}{\sqrt{1-\cos^{2m} y}} \sin(\alpha^q \cos y) dy \\
&= \frac{\pi}{2q}\alpha^{p+1} \int_0^{1} \cos^{(p+2-q)/q} \left(\frac{\pi}{2}r\right) \sqrt{\frac{1-\cos^2\left(\frac{\pi}{2}r\right)}{1-\cos^{2m}\left(\frac{\pi}{2}r\right)}} \\
&\qquad \times \sin\left(\alpha^q \cos\left(\frac{\pi}{2}r\right)\right) dr.
\end{aligned}
$$

Key Lemma

We put

$$f(r) = \cos^{(p+2-q)/q}\left(\frac{\pi}{2}r\right)\sqrt{\frac{1-\cos^2\left(\frac{\pi}{2}r\right)}{1-\cos^{2m}\left(\frac{\pi}{2}r\right)}}, \quad \mu = \alpha^q \qquad (5.9)$$

and $h(r) = \cos(\pi r/2)$ in (5.5). We note that $f(0) = \sqrt{q}$.

(i) **If** $f \in C^2[0,1]$, then by (5.5) and (5.8), we obtain

$$K(\alpha) = \sqrt{\frac{\pi}{2q}}\alpha^{p+1-q/2}\sin\left(\alpha^q - \frac{\pi}{4}\right) + o\left(\alpha^{p+1-q/2}\right). \qquad (5.10)$$

This implies our conclusion (5.6).

(ii) **Finally, we consider the case** $f \notin C^2[0,1]$**.** For instance, if $q > (p+2)/3$, then $\cos^{(p+2-q)/q}\left(\frac{\pi}{2}r\right) \notin C^2[0,1]$. Fortunately, we are still able to apply Lemma 5.1 to this case by modifying the proof of Lemma 5.1, and obtain (4.5). Thus the proof is complete. □

Now Theorem 4.1 follows from (5.2) and Lemma 5.2. □

$\int_0^1 f(r)e^{i\mu h(r)}dr = e^{i(\mu-(\pi/4))}\sqrt{\frac{2}{\pi\mu}}f(0) + O\left(\frac{1}{\mu}\right)$

For completeness, we show that (5.4) holds. Recall that $h(r) = \cos(\pi r/2)$, $0 \le p < 1$ and $0 < q \le 1$. For $m = 1/q$ and $0 \le x \le 1$, we put

$$f(x) = f_1(x)f_2(x) := \cos^{(p+2-q)/q}\left(\frac{\pi}{2}x\right)\sqrt{\frac{1-\cos^2\left(\frac{\pi}{2}x\right)}{1-\cos^{2m}\left(\frac{\pi}{2}x\right)}}. \quad (5.11)$$

(i) By direct calculation, we can show that if $q > 0$, namely, $m > 1$, then $f_2(x) \in C^2[0,1]$.

(ii) The essential point of the proof of (5.4) is to show that, as $\mu \to \infty$,

$$\Phi(\mu) := \int_0^1 f(x)e^{-i\mu x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i(\pi/4)}f(0) + O\left(\frac{1}{\mu}\right). \quad (5.12)$$

We put

$$w(x) := \frac{f(x)-f(0)}{x}, \quad \text{namely } f(x) = f(0) + xw(x).$$

By [10, Lemma 2.24],

$$\int_0^1 e^{-i\mu x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i\pi/4} + O\left(\frac{1}{\mu}\right). \tag{5.13}$$

Since $f(0) = \sqrt{q}$, by (5.13), we obtain

$$\begin{aligned} \Phi(\mu) &= f(0)\int_0^1 e^{-i\mu x^2}dx + \int_0^1 xe^{-i\mu x^2}w(x)dx \quad (5.14) \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i\pi/4}\sqrt{q} + O\left(\frac{1}{\mu}\right) + \int_0^1 xe^{-i\mu x^2}w(x)dx. \end{aligned}$$

We put

$$\Phi_1(\mu) := \int_0^1 xe^{-i\mu x^2}w(x)dx. \tag{5.15}$$

Now we prove that $w(x) \in C^1[0,1]$, because if it is proved, then by integration by parts, we easily show that $\Phi_1(\mu) = O(1/\mu)$ and our conclusion (5.4) follows immediately from (5.12) and (5.14). To do this, there are several cases to consider.

・We note that, by direct calculation, we can show that **if $q > 0$, namely, $m > 1$, then $f_2(x) \in C^2[0,1]$.**

Case 1. Assume that $p = 0$ and $q = 1$. Then $f(x) = \cos\left(\frac{\pi}{2}x\right) \in C^2[0,1]$.

Case 2. Assume that $0 < q < 1$ and $p + 2 \geq 3q$. Then $(p + 2 - q)/q \geq 2$ and $f_1(x) \in C^2[0,1]$. Consequently, $f \in C^2[0,1]$ in this case.

Case 3. Assume that $0 < p < 1$ and $q = 1$. Then $f(x) = \cos^{p+1}\left(\frac{\pi}{2}x\right) \notin C^2[0,1]$. However, by direct calculation, we can show that

$$w(x) = \frac{f(x) - f(0)}{x} \in C^1[0,1].$$

It is reasonable, because by Taylor expansion, for $0 < x \ll 1$, we have

$$w(x) = -\frac{(p+1)\pi^2}{8}x + o(x). \tag{5.16}$$

Case 4. Assume that $0 < q < 1$ and $p + 2 < 3q$. Then

$$\frac{p+2-q}{q} = \frac{p+2-2q}{q} + 1 := \eta + 1.$$

Then $0 < \eta < 1$ and $f_1(x) = \cos^{\eta+1} x$. Since $f_2 \in C^2[0,1]$, by the consequence of Case 3 above, we find that $w \in C^1[0,1]$. Thus the proof is complete. □

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Thank you very much

Thank You for Your Attention