Nonlinear elliptic singular perturbation problems on compact metric graphs

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1 Introduction on Nonlinear PDEs on compact metric graphs

Recently, mathematical studies of nonlinear PDEs on compact or non-compact metric graphs have been done in various topics, e.g. the mathematical analysis of the existence and stability of the ground state for the nonlinear Schrödinger equation ([AST], [KNP] and the references therein) and the studies the existence and the asymptotic behavior of solutions for reaction-diffusion equations or systems ([Y], [JM], [DLPZ], [CC] and the references therein).

In this talk, we present recent studies on the effect of metric graphs on the location of peaks of stationary solutions for some singular perturbed variational problem and for Schnakenberg model, one of the pattern formation model which describe autocatalytic phenomena in chemical reaction.

Basic Question: *How the network structure of the metric graph $\mathcal{G}$ affects the structure of solutions to PDEs on $\mathcal{G}$?*

The first part of this talk is based on a joint work with Prof. Masakata Shibata (Meijo Univ.) and the second part is based on a joint work with Prof. Yuta Ishii (National Institute of Technology, Ibaraki College).

1.1 Sobolev spaces on compact metric graphs

Let $\mathcal{G} := (E, V)$ is a connected and compact metric graph, which consists of the set of edges $E := \{e_j\}_{j=1}^N$ and the set of vertices V. Each edge $e \in E$ can be written as $e = \{v, v'\}$ with associated vertices v and v'. For each edge $e = \{v, v'\} \in E$, with $v, v' \in V$, we identify $e = \{v, v'\}$ with the interval $[0, l(e)]$ and $v = 0, v' = l(e)$ with a local coordinate, where $l(e)$ is the length of the edge. We denote

$$V_{int} := \{v \in V \mid \deg(v) \geq 3\}, \quad V_{end} := \{v \in V \mid \deg(v) = 1\}.$$

We may assume that $V = V_{int} \cup V_{end}$. In some case, we assume that the graph has no self-loop, namely each edge $e = \{v, v'\} \in E$ has different vertices $(v \neq v')$.

Now, we introduce Sobolev spaces on a compact metric graph $\mathcal{G} = (V, E)$. For $l = 0, 1, 2$, define

$$H^l(\mathcal{G}) := \{u \in C(\mathcal{G}) \mid u^{(e)} := u|_e \in H^l(e) \ (\forall e \in E)\}$$

and $L^2(\mathcal{G}) := H^0(\mathcal{G})$. Although one can consider complex-valued functions, in this talk we only consider real-valued function on the metric graphs. In particular, $H^1(\mathcal{G})$ is a Hilbert space with the inner product:

$$(u, v)_{H^1(\mathcal{G})} := \int_{\mathcal{G}} uv\, dx + \int_{\mathcal{G}} u'v'\, dx = \sum_{j=1}^{N} \int_{e_j} \Big(u^{(e)}(x)v^{(e)}(x) + (u^{(e)})'(x)(v^{(e)})'(x) \Big)\, dx$$

and the associated norm:

$$\|u\|_{H^1(\mathcal{G})} = \sqrt{(u,u)_{H^1(\mathcal{G})}} = \left(\int_{\mathcal{G}} (|u'|^2 + |u|^2)\, dx \right)^{\frac{1}{2}}.$$

It is well-known that on an interval I a function $u \in H^1(I)$ can be identified with a continuous function on I and satisfies the formula of the integration by parts. Since $u \in H^1(\mathcal{G})$ is continuous, u still satisfies the formula of the integration by parts. Thus we have the following estimate for $u \in H^1(\mathcal{G})$:

$$|u(P) - u(Q)| \le d_{\mathcal{G}}(P,Q)^{\frac{1}{2}} \|u'\|_{L^2(\mathcal{G})}^{\frac{1}{2}} \quad (\forall P, Q \in \mathcal{G}),$$

where $d_{\mathcal{G}}(P,Q)$ is the natural distance on $\mathcal{G}$ between P and Q. Also, there exists a constant C such that

$$\|u\|_{L^\infty(\mathcal{G})} \le C\|u\|_{H^1(\mathcal{G})}.$$

So, as in the interval case, we have the compact embedding $H^1(\mathcal{G}) \subset C(\mathcal{G})$. We also consider the Sobolev space

$$H_0^1(\mathcal{G}) := \{u \in H^1(\mathcal{G}) \mid u^{(e)}(v) = 0 \ (\forall v \in V_{ext} \text{ such that } e \succ v)\}.$$

1.2 nonlinear elliptic PDEs on compact metric graphs

We briefly explain a weak solution u for a nonlinear elliptic equation $-du'' = g(x,u)$ on a compact metric graph $G = (V,E)$ under several boundary conditions, where $d > 0$ is a diffusion constant and $g(x,t)$ is a certain given nonlinearity. If $u \in H^1(\mathcal{G})$ satisfies

$$d \int_{\mathcal{G}} u'\varphi'\, dx = \int_{\mathcal{G}} g(x,u(x))\varphi(x)\, dx \quad (\forall \varphi \in H^1(\mathcal{G})),$$

we say that $u \in H^1(\mathcal{G})$ is a (Neumann-)weak solution to the problem:

$$-du''(x) = g(x,u(x)) \quad (\forall e \in E),$$

$$\sum_{e \succ v} \partial u^{(e)}(v) = 0 \quad (\forall v \in V_{int}),$$

and

$$\partial u^{(e)}(v) = 0 \quad (\forall v \in V_{ext} \text{ such that } e \succ v).$$

Here, $e \succ v$ means that e is incident to $v \in V$ and $\partial u^{(e)}(v)$ is the outward derivative of $u^{(e)}$ at v. Note that the condition $\sum_{e \succ v} \partial u^{(e)}(v) = 0 \quad (\forall v \in V_{int})$ is called the Kirchhoff condition which naturally appears at $v \in V_{int}$ as in the Neumann boundary condition. Actually, for a nice nonlinearity $g(x,t)$, we can see that a weak solution $u \in H^1(\mathcal{G})$ belongs to $H^2(\mathcal{G})$ and satisfies the Kirchhoff conditions at vertices $v \in V_{int}$ and the Neumann boundary condition at vertices $v \in V_{ext}$. In a similar way, if $u \in H_0^1(\mathcal{G})$ satisfies

$$d \int_{\mathcal{G}} u'\varphi' \, dx = \int_{\mathcal{G}} g(x, u(x))\varphi(x) \, dx \quad (\forall \varphi \in H_0^1(\mathcal{G})),$$

we say that $u \in H_0^1(\mathcal{G})$ is a (Dirichlet-)weak solution to the problem:

$$-du''(x) = g(x, u(x)) \quad (\forall e \in E),$$

$$\sum_{e \succ v} \partial u^{(e)}(v) = 0 \quad (\forall v \in V_{int}),$$

and

$$u^{(e)}(v) = 0 \quad (\forall v \in V_{ext} \text{ such that } e \succ v).$$

When $d > 0$ is small, we call these problems as singular perturbed nonlinear elliptic problems. Solutions for the singular perturbed problems are often localized and have a spiky profile at several points.

We use the notation:

$H^2_{KN}(\mathcal{G}) := \{u \in H^2(\mathcal{G}) \mid u$ satisfies the Kirchhoff conditions at $v \in V_{int}$ and the Neumann conditions at $v \in V_{ext}\}$,

$H^2_{KD}(\mathcal{G}) := \{u \in H^2(\mathcal{G}) \mid u$ satisfies the Kirchhoff conditions at $v \in V_{int}$ and the Dirichlet conditions at $v \in V_{ext}\}$.

2 Least energy solution to a singularly perturbed variational problem

Consider positive solutions to the following problem with small constant $\epsilon > 0$:

$$-\epsilon^2 u''(x) + u(x) = f(u(x)) \quad (x \in \mathcal{G}), \quad u(x) > 0 \quad (x \in \mathcal{G}), \tag{1}$$

$$\sum_{e \succ v} \partial u(v) = 0 \quad (v \in V_{int}), \tag{2}$$

When $V_{end} \neq \emptyset$, we also impose Neumann boundary condition:

$$\partial u(v) = 0 \quad (\forall v \in V_{end})$$

or Dirichlet boundary condition

$$u(v) = 0 \quad (\forall v \in V_{end}),$$

respectively. Typical nonlinearity is as follows:

$$f(t) := |t|^{p-1}t \ (1 < p < +\infty).$$

For $u \in H^1(\mathcal{G})$, we define the energy:

$$J_\epsilon(u) := \frac{\epsilon^2}{2} \int_{\mathcal{G}} |u'(x)|^2 \, dx + \int_{\mathcal{G}} u(x)^2 \, dx - \int_{\mathcal{G}} F(u(x)) \, dx,$$

where $F(t) := \int_0^t f(s) \, ds$. In particular, when $f(t) = |t|^{p-1}t$, we have

$$J_\epsilon(u) := \frac{\epsilon^2}{2} \int_{\mathcal{G}} |u'(x)|^2 \, dx + \int_{\mathcal{G}} u(x)^2 \, dx - \frac{1}{p+1} \int_{\mathcal{G}} |u(x)|^{p+1} \, dx,$$

Define

$$\sigma_\epsilon(\mathcal{G}) := \inf_{u \in H^1(\mathcal{G}), u \neq 0} \left(\sup_{t>0} J_\epsilon(tu) \right).$$

Proposition 2.1 *Fix $\epsilon > 0$. There exists a least energy solution u_ϵ which satisfies $\sigma_\epsilon(\mathcal{G}) = J_\epsilon(u_\epsilon)$ and has the least energy among all nontrivial solutions.*

Remark 1 *We note the following characterization:*

$$\sigma_\epsilon(\mathcal{G}) = \inf_{w \in N_\epsilon(\mathcal{G}} J_\epsilon(w),$$

where

$$N_\epsilon(\mathcal{G}) := \left\{ w \neq 0, w \in H^1(\mathcal{G}) \mid \epsilon^2 \int_{\mathcal{G}} |w'|^2 \, dx + \int_{\mathcal{G}} |w|^2 \, dx = \int_{\mathcal{G}} |w|^{p+1} \, dx \right\}.$$

$N_\epsilon(\mathcal{G})$ is called the Nehari manifold and its is easy to see that all non-trivial solutions belongs to $N_\epsilon(\mathcal{G})$. If $f(t) = |t|^{p-1}t$, u_ϵ is also obtained as a minimizer to the following variational problem:

$$\Sigma_\epsilon(\mathcal{G}) := \inf \left\{ \frac{\epsilon^2 \int_{\mathcal{G}} |v'|^2 \, dx + \int_{\mathcal{G}} |v|^2 \, dx}{\left(\int_{\mathcal{G}} |v|^{p+1} \, dx \right)^{\frac{2}{p+1}}} \mid v \neq 0, v \in H^1(\mathcal{G}) \right\}.$$

Actually, we have the relation:

$$\sigma_\epsilon(\mathcal{G}) = (\frac{1}{2} - \frac{1}{p+1}) \Sigma_\epsilon(\mathcal{G})^{\frac{p+1}{p-1}}.$$

(proof.) The proof is standard. In particular, for the case $f(t) := |t|^{p-1}t$, by using the remarks above and the compact embedding $H^1(\mathcal{G}) \subset C(\mathcal{G})$, it is easy to see the existence of the non-negative minimizer $v_\epsilon (\neq 0)$ to $\Sigma_\epsilon(\mathcal{G})$. Since the strong maximum principle also holds on the metric graphs, we can conclude $v_\epsilon(x) > 0$ $(x \in \mathcal{G})$. For certain $t_\epsilon > 0$, $u_\epsilon(x) = t_\epsilon v_\epsilon(x)$ is a minimizer to σ_ϵ. This completes the proof.

Question: Effect of the geometry of $\mathcal{G}$ on the asymptotic shape of u_ϵ and the location of the maximum point x_ϵ of u_ϵ as $\epsilon \to 0$.

2.1 main results

For simplicity, we assume $f(t) = |t|^{p-1}t$ with $1 < p < +\infty$. We state main results in [KS].

Theorem 1 (Neumann problem) *Suppose $V_{int} \neq \emptyset$ and $V_{end} \neq \emptyset$. Then, for sufficiently small $\epsilon > 0$ we have the followings:*
(1) $x_\epsilon \in V_{end}$.
(2) Let $x_\epsilon \in e \in E$ and idenfify $e = [0, l(e)]$ with $x_\epsilon = 0$. Then, $u_\epsilon(\epsilon x) \to \Phi(x)$ in $C^2_{loc}([0, +\infty)$, where

$$-\Phi'' + \Phi = f(\Phi),\ \Phi(x) > 0\ (x \in \mathbb{R}),\quad \Phi(0) = \max \Phi,\ \Phi(x) \to 0\ (|x| \to \infty).$$

(3) e is the longest edge in $E_{end} := \{e \in E \mid e \succ v \in V_{end}\}$, i.e.

$$l(e) = \max_{e' \in E_{end}} l(e')(:= l^*_N).$$

(4) The asymptotic behavior of the energy:

$$\sigma_\epsilon = \epsilon \left\{ \frac{\sigma}{2} + \exp\left(-\frac{2l^*_N}{\epsilon}(1 + o(1)) \right) \right\} \quad (\epsilon \to 0).$$

Here, $\sigma > 0$ is the least energy associated with Φ on $\mathbb{R}$.

Theorem 2 (Dirichlet problem) *Suppose $V_{int} \neq \emptyset$ and $\mathcal{G}$ has no self-loop. (may $V_{end} = \emptyset$). Then, for sufficiently small $\epsilon > 0$ we have the followings:*
(1) $x_\epsilon \in \mathcal{G} \setminus V$.
(2) Let $x_\epsilon \in e \in E$ and idenfify $e = [0, l(e)]$. Then $x_\epsilon \to \frac{l(e)}{2}$. Then, $u_\epsilon(x_\epsilon + \epsilon x) \to \Phi(x)$ in $C^2_{loc}(\mathbb{R})$. (3) e is the longest edge in E, i.e.

$$l(e) = \max_{e' \in E} l(e')(:= l^*_D).$$

(4) The asymptotic behavior of the energy:

$$\sigma_\epsilon = \epsilon \left\{ \sigma + \exp\left(-\frac{l^*_D}{\epsilon}(1 + o(1)) \right) \right\} \quad (\epsilon \to 0).$$

•Related results:

(1) Ni and Takagi [NT1, NT2] studied the same Neumann problem on a domain $\Omega \subset \mathbb{R}^N$. When $N \geq 2$, the least energy solution concentrates near the point $P \in \partial\Omega$ which has the maximum mean curvature.
(2) Ni and Wei [Ni-Wei] studied the same Dirichlet problem on a domain $\Omega \subset \mathbb{R}^N$. When $N \geq 2$, the least energy solution concentrates near the point $P \in \partial\Omega$ which has the maximum distance to $\partial\Omega$.

(3) Dovetta et al (2020) [DGMP] studied essentially same Neumann problem:

$$-u'' + \lambda u = u^p \quad (x \in \mathcal{G})$$

with large $\lambda > 0$ on metric graphs. However, their result does not give the precise location of the maximum point $x_\epsilon \in V$ and the precise asymptotic expansion of the energy σ_ϵ. They also construct one-peak solution which concentrates at any vertex $v \in V$ and multi-peak solutions which concentrate at several vertices by using the Lyapunov-Schmidt method.
(4) In [BMP] they proved similar results for the special case $f(t) = |t|^2 t$ by using elliptic functions. Our results can be applied fro more general nonlinearity.
(5) Shibata [Shi2] also constructed a solution v_ϵ which concentrates near each vertex $v \in V_{int}$ with deg$v = 3$ such that $J_\epsilon(v_\epsilon) = \epsilon(\frac{3}{2}\sigma + o(1))$.

2.2 Sketch of the proof

We only give a sketch of the proof of Theorem 1.

Let $\overline{u}(y) := u(\epsilon y)$ for $y \in \mathcal{G}_\epsilon := \frac{1}{\epsilon}\mathcal{G}$, which is the rescaled graph w.r.t. the origin identifying the edge $e \in E$ with $[0, l(e)]$. Then, we have

$$J_\epsilon(u) = \epsilon I_\epsilon(\overline{u}),$$

where

$$I_\epsilon(\overline{u}) = I(\overline{u}; \mathcal{G}_\epsilon) := \frac{1}{2}\int_{\mathcal{G}_\epsilon} (\overline{u}'(y))^2 + (\overline{u}(y))^2\, dy - \frac{1}{p+1}\int_{\mathcal{G}_\epsilon} |\overline{u}(y)|^{p+1}\, dy.$$

So,

$$\sigma_\epsilon = \epsilon\overline{\sigma}_\epsilon,$$

where

$$\overline{\sigma}_\epsilon = \inf_{v \in H^1(\mathcal{G}_\epsilon), v \neq 0}\left(\sup_{t>0} I(tv; \mathcal{G}_\epsilon)\right) = I(\overline{u_\epsilon}).$$

2.2.1 Upper bound

Take $\hat{e_*} = [0, \frac{l_N^*}{\epsilon}]$. Write $l^* := l_N^*$, for simplicity. Consider the function $w_\epsilon \in H^1(\mathcal{G}_\epsilon)$ as follows:

$$w_\epsilon(y) := \begin{cases} \Phi(y) & (0 \le y \le \frac{l^*}{\epsilon} - 1) \\ \Phi(\frac{l^*}{\epsilon} - 1)(\frac{l^*}{\epsilon} - y) & (\frac{l^*}{\epsilon} - 1 \le y \le \frac{l^*}{\epsilon}) \\ 0 & (y \in \mathcal{G}_\epsilon \setminus \hat{e_*}). \end{cases}$$

Then, we have

$$\begin{aligned} \overline{\sigma}_\epsilon &\le \sup_{t>0} I(tw_\epsilon; \mathcal{G}_\epsilon) = \sup_{t>0} I(tw_\epsilon; [0, \frac{l^*}{\epsilon}]) = I(t_\epsilon w_\epsilon; [0, \frac{l^*}{\epsilon}]) \\ &= I(t_\epsilon \Phi; [0, \frac{l^*}{\epsilon} - 1]) + I(t_\epsilon w_\epsilon; [\frac{l^*}{\epsilon} - 1, \frac{l^*}{\epsilon}]) \end{aligned}$$

for some $t_\epsilon > 0$. Note $t_\epsilon = 1 + o(1)$, since Φ is the least energy solution on $[0, +\infty)$. Here, note that

$$\begin{aligned} I(t_\epsilon \Phi; [0, \frac{l^*}{\epsilon} - 1]) &= I(t_\epsilon \Phi; [0, +\infty)) - I(t_\epsilon \Phi; [\frac{l^*}{\epsilon} - 1, +\infty)) \\ &\leq I(t_\epsilon \Phi; [0, +\infty)) \leq \sup_{t>0} I(t\Phi; [0, +\infty)) = \frac{\sigma}{2}. \end{aligned}$$

Here, we used $I(t_\epsilon \Phi; [\frac{l^*}{\epsilon} - 1, +\infty)) \geq 0$, since Φ is small enough on $[\frac{l^*}{\epsilon} - 1, +\infty)$. On the other hand, it follows

$$\begin{aligned} I(t_\epsilon w_\epsilon; [\frac{l^*}{\epsilon} - 1, \frac{l^*}{\epsilon}]) &\leq \frac{t_\epsilon^2}{2} \int_{\frac{l^*}{\epsilon}-1}^{\frac{l^*}{\epsilon}} (w_\epsilon')^2 + (w_\epsilon)^2 \\ &= \frac{2}{3} t_\epsilon^2 \Phi^2(\frac{l^*}{\epsilon} - 1) \leq \Phi^2(\frac{l^*}{\epsilon} - 1). \end{aligned}$$

Thus, we obtain

$$\overline{\sigma_\epsilon} \leq \frac{\sigma}{2} + \Phi^2(\frac{l^*}{\epsilon} - 1).$$

Here, we note $\Phi(x) = \exp(-x(1 + o(1)))$ as $x \to +\infty$. Therefore, we conclude

$$\overline{\sigma_\epsilon} \leq \frac{\sigma}{2} + \exp(-\frac{2l^*}{\epsilon}(1 + o(1))).$$

2.2.2 Lower bound

First, we can show that $_\epsilon$ has its maximum at some $x_\epsilon = v \in V$. Actually, this follows from the rough upper bound $\overline{\sigma_\epsilon} \leq \frac{\sigma}{2} + o(1)$.

Let $x_\epsilon = v \in V \in \hat{e_1}$ with $\hat{e_1} = [0, \frac{l(e_1)}{\epsilon}]$ with the identification $x_\epsilon = 0$. Write $l := l(e_1)$, for simplicity. Then, we want to claim that

Claim: We have

$$\overline{\sigma_\epsilon} \geq \frac{\sigma}{2} + \exp(-\frac{2l}{2}(1 + o(1))).$$

Let $\partial \hat{e} = \{v_1, v_0\}$ with $v_1 := x_\epsilon \in V_{end}$, $v_0 \in V_{int}$ and let $k := \deg(v_0) \geq 3$. Let $\{\hat{e}_i\}_{i=1}^k = \{\hat{e} \in \hat{E} \mid \hat{e} \succ v_0\}$.

Decompose $\mathcal{G}_\epsilon$ into $\mathcal{G}_\epsilon' := (\{v_1, v_0\}, \hat{e_1})$ and $\mathcal{G}_\epsilon'' := (V \setminus \{v_1\}, \hat{E} \setminus \{\hat{e_1}\})$.

Now,

$$\overline{\sigma_\epsilon} = \sup_{t>0} I(t\overline{u}_\epsilon) \geq I(t\overline{u}_\epsilon; \mathcal{G}_\epsilon') + I(t\overline{u}_\epsilon; \mathcal{G}_\epsilon'')$$

for any $t > 0$. Define $v_\epsilon \in H^1(0, +\infty)$ such that

$$v_\epsilon(y) = \begin{cases} \overline{u}_\epsilon(y) & (0 \leq y \leq \frac{l}{\epsilon}) \\ m_\epsilon e^{\frac{l}{\epsilon} - y} & (\frac{l}{\epsilon} \leq y < +\infty) \end{cases}$$

where $m_\epsilon := \overline{u}_\epsilon(\frac{l}{\epsilon}) = \overline{u}_\epsilon(v_0)$. Choose $t_\epsilon > 0$ so that

$$I(t_\epsilon v_\epsilon; [0, +\infty)) = \sup_{t>0} I(tv_\epsilon; [0, +\infty)) \geq \frac{\sigma}{2}.$$

Note that $t_\epsilon = 1 + o(1)$, since we can show $v_\epsilon \to \Phi$. Now,

$$I(t_\epsilon \overline{u}_\epsilon; \mathcal{G}'_\epsilon) = I(t_\epsilon v_\epsilon; [0, +\infty)) - I(t_\epsilon v_\epsilon; [\frac{l}{\epsilon}, +\infty)).$$

So, we have

$$\overline{\sigma}_\epsilon \geq \frac{\sigma}{2} - I(t_\epsilon v_\epsilon; [\frac{l}{\epsilon}, +\infty)) + I(t\overline{u}_\epsilon; \mathcal{G}''_\epsilon).$$

Here,

$$I(t_\epsilon v_\epsilon; [\frac{l}{\epsilon}, +\infty)) \leq \frac{t_\epsilon^2}{2} \int_{\frac{l}{\epsilon}}^{\infty} (v'_\epsilon)^2 + (v_\epsilon)^2 = \frac{t_\epsilon^2}{2} m_\epsilon^2 = \frac{1}{2} m_\epsilon^2 (1 + o(1)).$$

Thus, it follows

$$\overline{\sigma}_\epsilon \geq \frac{\sigma}{2} - \frac{1}{2} m_\epsilon^2 (1 + o(1)) + I(t\overline{u}_\epsilon; \mathcal{G}''_\epsilon).$$

First, since $\|\overline{u_\epsilon}\|_{L^\infty(\mathcal{G}''_\epsilon)} = m_\epsilon = o(1)$, we have

$$\begin{aligned}
I(t\overline{u}_\epsilon; \mathcal{G}''_\epsilon) &= \frac{t_\epsilon^2}{2} \int_{\mathcal{G}''_\epsilon} (\overline{u_\epsilon}')^2 + (\overline{u_\epsilon})^2 \, dx - \frac{t_\epsilon^{p+1}}{p+1} \int_{\mathcal{G}''_\epsilon} |\overline{u_\epsilon}|^{p+1} \, dx \\
&= \frac{1}{2}(1 + o(1)) \int_{\mathcal{G}''_\epsilon} (\overline{u_\epsilon}')^2 + (\overline{u_\epsilon})^2 \, dx \\
&= \frac{1}{2}(1 + o(1)) \int_{\mathcal{G}''_\epsilon} (\overline{u_\epsilon u_\epsilon}')' + |\overline{u_\epsilon}|^{p+1} \, dx \\
&\geq \frac{1}{2}(1 + o(1)) \int_{\mathcal{G}''_\epsilon} (\overline{u_\epsilon u_\epsilon}')' \\
&= \frac{1}{2}(1 + o(1)) \left(\sum_{i=2}^{k} (\partial \overline{u}_\epsilon|_{\hat{e_i}}(v_0)) \overline{u}_\epsilon(v_0) \right).
\end{aligned}$$

Claim 1: For each $i = 2, 3, \cdots, k$, we have

$$(\partial \overline{u}_\epsilon|_{\hat{e_i}}(v_0)) \overline{u}_\epsilon(v_0) \Big) = m_\epsilon^2 (1 + o(1)).$$

It follows

$$\overline{\sigma}_\epsilon \geq \frac{\sigma}{2} + \frac{k-2}{2} m_\epsilon^2 (1 + o(1)).$$

Claim 2:

$$m_\epsilon = \overline{u}_\epsilon(v_0) \geq \frac{2M_0}{k+1} e^{-\frac{l}{\epsilon}} (1 + o(1)),$$

where $M_0 = \Phi(0)$.
(proof.) Consider

$$-z'' + z = 0 \text{ on } (0, \frac{l}{\epsilon}),$$

with $z(0) = \overline{u}_\epsilon(0), z'(\frac{l}{\epsilon}) + kz(\frac{l}{\epsilon}) = 0$. By the Kirchhoff condition at v_0 and a comparison theorem, we have

$$\overline{u}_\epsilon(y) \geq z(y) \quad (y \in (0, \frac{l}{\epsilon})).$$

Thus, by using the explicit expression of $z(y)$, we get

$$\begin{aligned} m_\epsilon &= \overline{u}_\epsilon = \overline{u}_\epsilon(v_0) \\ &\geq \frac{2\overline{u}_\epsilon(0)}{(k+1)e^{\frac{l}{\epsilon}} - (k-1)e^{-\frac{l}{\epsilon}}} = \frac{2}{k+1}e^{-\frac{l}{\epsilon}}(\Phi(0)+o(1))(1+o(1)) \\ &= \frac{2M_0}{k+1}e^{-\frac{l}{\epsilon}}(1+o(1)). \end{aligned}$$

Therefore,

$$\overline{\sigma}_\epsilon \geq \frac{\sigma}{2} + \frac{k-2}{2}(\frac{2M_0}{k+1})^2 e^{-\frac{2l}{\epsilon}}(1+o(1)) = \frac{\sigma}{2} + \exp(-\frac{2l}{\epsilon}(1+o(1))).$$

Remark 2 *Let $E'_{end} := \{e \in E_{end} \mid l(e) = l^*_N\}$. If the number of E'_{end} is greater than two, what happens? Recently, when $f(t) = |t|^{p-1}t$ with $p > 1$, Shibata [S1] answered in the following way: Let* deg $v(e^*)$:= *the smallest number of* deg $v(e)$ *among $e \in E'_{end}$, where $v(e) \in V_{int}$. Then, he proved*

$$\overline{\sigma}_\epsilon = \frac{\sigma}{2} + C_p \frac{\deg\ v(e^*) - 2}{\deg\ v(e^*)} \exp\left(-\frac{2l^*_N}{\epsilon}\right)(1+o(1)).$$

So, the least energy solution u_ϵ concentrates on $v(e^*)$.

3 Spiky stationary solutions to the Schnakenberg model with heterogeneity

Consider positive stationary solutions to

$$u_t = \epsilon^2 u_{xx} - u + g(x)u^2 v, \quad \epsilon v_t = Dv_{xx} + \frac{1}{L} - \frac{cg(x)}{\epsilon}u^2 v, \quad (x \in \mathcal{G}, t > 0), \tag{3}$$

$$\sum_{e \succ \mathrm{v}} \partial u^{(e)}(\mathrm{v}) = 0, \quad \sum_{e \succ \mathrm{v}} \partial v^{(e)}(\mathrm{v}) = 0 \quad (\mathrm{v} \in V). \tag{4}$$

Here, $L := |\mathcal{G}|$ be the total length of $\mathcal{G}$, $\epsilon > 0, D > 0, c > 0$ are constants, $g(x)$ is a positive continuous function on $\mathcal{G}$ with $g \in C^3(e)$ for each $e \in E$.

In the case $\mathcal{G} = (V, E)$ with $E = \{e\}, e = [-1, 1]$, $V = \{\pm 1\}$, $g(x) = 1$, in the seminal paper [IWW], Iron, Wei and Winter constructed a solution $U_\epsilon, v_\epsilon)$ such that

$$u_\epsilon(x) \sim \frac{1}{6c}w(\frac{x}{\epsilon}), \quad v_\epsilon(0) \sim 6c$$

for small $\epsilon > 0$ and studied its stability. Here, $w(x)$ is the ground state solution to $-w'' + w = w^2$ on $\mathbb{R}$ with $w(0) = \max w$. They also constructed multi-peak symmetric solutions and studied their stability.

Question: Can we construct such solutions on a given metric graph? How the geometry of the metric graph determines the location of the concentration points?

We give an answer to this question. We will explain that the geometry of the metric graphs affects the location of concentration points through the associated Green function on the metric graph.

3.1 The Green function for the metric graph

We introduce the Green function on a metric graph $\mathcal{G}$, which plays an important role in our analysis.

For $\int_{\mathcal{G}} f\,dx = 0$, a solution η of the following problem:

$$D\eta''(x) = f(x) \quad (x \in \mathcal{G}), \quad \sum_{e \succ \mathrm{v}} \partial\eta(\mathrm{v}) = 0 \ (\mathrm{v} \in V)$$

can be express as

$$\eta(x) - \frac{1}{L}\int_{\mathcal{G}} \eta(s)\,ds = \int_{\mathcal{G}} G(x,s) f(s)\,ds.$$

We also impose $\int_{\mathcal{G}} G(x,s)\,ds = 0$. The function $G(x,s)$ is the Green function associated with the metric graph $\mathcal{G}$ which plays an important role in our study.

We assume some conditions on $G(x,s)$. In particular, there exist functions $m_{ij}(t), t = (t_1, t_j)$ and $K_{ij}(y,z)$ such that

$$G(y + t_i, z + t_j) - G(t_i, z + t_j) = m_{ij}(t)y + K_{i,j}(y,z) \tag{5}$$

for $t_i, y + t_i \in e_i$ and $t_j, z + t_j \in e_j$ with the following properties:

- $|K_{ij}| = O(|y|)$;
- $\int_{-r}^{r} K_{ij}(y,z)P(z)\,dz$ is an even function in y, where $P(z)$ is an even function in z for small $r > 0$.

3.1.1 The Green function for Y-shaped graph

For the Green function for Y-shapede graph, we have

Lemma 3.1 *For $x \in e_j$, we have*

$$\begin{aligned} G(x,s) &= \frac{1}{D}\left\{\frac{1}{2}\Big[|x-s| - (x+s)\Big]\chi_{e_j}(s) - \frac{1}{2L}(x - l_j)^2 + \frac{l_j^2}{2L}\right. \\ &- \frac{1}{2L}\sum_{k-1}^{3}(s-l_k)^2\chi_{e_k}(s) + \frac{1}{2L}\sum_{k=1}^{3} l_k^2\chi_{e_k}(s) - \frac{1}{3L^2}\sum_{k=1}^{3} l_k^3. \end{aligned}$$

For the proof of this Lemma, see [KS1]. So, if $x, s \in e_1$, it follows

$$\begin{aligned} G(x,s) &= \frac{1}{D}\left\{\frac{1}{2}(|x-s| - (x+s)) - \frac{1}{2L}(x-l_1)^2 + \frac{l_1^2}{2L}\right. \\ &- \left.\frac{1}{2L}(s-l_1)^2 + \frac{1}{2L}l_1^2 - \frac{1}{3L^2}\sum_{k=1}^{k} l_k^3\right\}. \end{aligned}$$

Thus, we get

$$\begin{aligned}
G(y+t_1, z+t_1) - G(t_1, z+t_1) &= \frac{1}{D}\left\{\frac{1}{2}(|y-z|-|z|) - (\frac{1}{2} + \frac{t_1 - l_1}{L})y - \frac{1}{2L}y^2\right\} \\
&:= m_{11}(t)y + \frac{1}{2D}(|y-z|-|z|) + O(y^2) \\
&:= m_{11}(t)y + K_{11}(y,z) + O(y^2),
\end{aligned}$$

where

$$m_{11}(t) = -\frac{1}{D}(\frac{1}{2} + \frac{t_1 - l_1}{L}), \quad K_{11}(y,z) := \frac{1}{2D}(|y-z|-|z|).$$

Note that

$$\int_{-r}^{r} (|y-z|-|z|)(w(\frac{z}{\epsilon})\chi(z))^2 \, dz$$

in an even function in y.

3.2 Construction of one peak solution

Now, we state the abstract theorem on the construction of pne-peak solution under certain assumption the Green function.

Take any edge $e \in E$. Define the function $F(t)$ for $t \in e$ as follows:

$$F(t) := m(t) + \frac{6cg'(t)}{g(t)^2},$$

where $m(t) := m_{11}(t)$.

Theorem 3 ([IK1]) *Assume the condition (5). For some edge $e \in E$, we assume that there exists $t^0 \in$ (the interior point in e) such that*

$$F(t^0) = 0, \quad F'(t^0) \neq 0.$$

There exists a stationary solution (u_ϵ, v_ϵ) such that $|t_\epsilon - t^0| \leq C\epsilon^{\frac{3}{4}}$ and

$$u_\epsilon(x) = \frac{1}{6c} w(\frac{x - t_\epsilon}{\epsilon})\chi(\frac{x - t_\epsilon}{r_0}) + \phi_{t_\epsilon}(x) := w_{\epsilon, t_\epsilon}(x) + \phi_{t_\epsilon}(x),$$

with $\|\overline{\phi_{t_\epsilon}}\|_{H^2(\mathcal{G}_\epsilon)} \leq C_0\epsilon$, and

$$v_\epsilon(t_\epsilon) = \frac{6c}{g(t_\epsilon)} + O(\epsilon).$$

Here, $w(x)$ is the ground state solution to $-w'' + w = w^2$ on $\mathbb{R}$ with $w(0) = \max w$ and $\chi(x)$ is a suitable even cut-off function around $x = 0$.

When $g(x) = 1$ and $\mathcal{G}$ is the Y-shaped graph, i.e. $\mathcal{G} = (V, E), E = \{e_j\}_{j=1}^3, V = \{O\} \cup \{P_j\}_{j=1}^3$, we can say the precise location of the peak of u_ϵ.

Theorem 4 ([IK1]) *Let $g(x) \equiv 1$ and $\mathcal{G}$ be a Y-shaped graph. Assume on the length $l_1 := |e_1|$ of the edge e_1 satisfies $l_1 > \frac{L}{2}$. Let x_0 be the point which has the distance $\frac{L}{2}$ from the boundary $\partial\mathcal{G}$. Then, for sufficiently small $\epsilon > 0$ there exists a solution (u_ϵ, v_ϵ) such that:*

$$u_\epsilon(x) = \frac{1}{6c} w(\frac{x - x_\epsilon}{\epsilon})\chi(x) + \phi_\epsilon(x), \quad \|\overline{\phi_\epsilon}\|_{H^2(\mathcal{G}_\epsilon)} \le C\epsilon, \quad v_\epsilon(x_\epsilon) = 6c + O(\epsilon).$$

Here, $|x_\epsilon - x_0| \le C\epsilon^{\frac{3}{4}}$.

Remark 3 *When $g(x) = 1$, Ishii proved that this solution is stabe for any $D > 0$. However, certain heterogeneity produces the threshould which destabilizes the one-peak solution (see Ishii-K. [IK2] and Ishii [I1,I2].)*

Remark 4 *Theorem says that the location of peak of one-peak solution is determined by non-local effect of the metric graph, in particular does not depend on the distance from the junction.*

Remark 5 *For each $x \in V_{ext}$, we can construct a boundary peak solution as an even solution for the symmetrically extended graph (see e.g. [IK2]).*

3.3 Heuristic argument on the approximated solution

Why we look for the solution like

$$u_\epsilon \sim \frac{1}{6c} w(\frac{x - x_\epsilon}{\epsilon}), \quad v_\epsilon(x_\epsilon) \sim \frac{6c}{g(x_\epsilon)}?$$

Assume that

$$u_\epsilon \sim Aw(\frac{x - x_\epsilon}{\epsilon})\chi(x), \quad v_\epsilon(x_\epsilon) \sim \xi.$$

From the second equation, we have

$$0 = -D \int_{\mathcal{G}} v''\, dx = \int_{\mathcal{G}} \left(\frac{1}{L} - \frac{c}{\epsilon} g u^2 v \right) dx = 1 - \frac{c}{\epsilon} \int_{\mathcal{G}} g u^2 v\, dx.$$

Now, define

$$\tilde{u}(y) := u(x_\epsilon + \epsilon y), \quad \tilde{v}(y) := v(x_\epsilon + \epsilon y), \quad \tilde{g}(y) := g(x_\epsilon + \epsilon y).$$

Then $y \in \tilde{\mathcal{G}}_\epsilon \sim \mathbf{R}$ and

$$0 = 1 - c \int_{\tilde{\mathcal{G}}_\epsilon} \tilde{g} \tilde{u}^2 \tilde{v}\, dy \sim 1 - c g(x_\epsilon) \int_{\mathbf{R}} A^2 w(y)^2 \xi\, dy.$$

Since, $\int_{\mathbf{R}} w^2(y)\, dy = 6$, it follows

$$1 = 6 c g(x_\epsilon) A^2 \xi.$$

On the other hand, we have

$$-\tilde{u}''(y) + \tilde{u}(y) = \tilde{g}(y)(\tilde{u}(y))^2\tilde{v}(y).$$

Thus, we have

$$-Aw''(y) + Aw(y) \sim g(x_\epsilon)A^2 w(y)^2 \xi^2.$$

Since, $-w'' + w = w^2$, it follows

$$1 = g(x_\epsilon)A\xi.$$

These relations imply

$$A = \frac{1}{6c}, \quad \xi = \frac{6c}{g(x_\epsilon)}.$$

3.4 How about two-peak solutions?

We present only a typical result for the Y-shaped graph.

In this case, we can construct two type of two-peak solutions. First, we consider the case in which two peaks locate on a different edges, e.g. e_1 and e_2. Assume $l_1 := |e_1| > \frac{L}{4}$, $l_2 := |e_2| > \frac{L}{4}$ and $l_1 = l_2$. We identify the junction v_0 as 0 on each edges. Under this identification, Let $t_1^0 := l_1 - \frac{L}{4} \in e_1$ and $t_2^0 := l_2 - \frac{L}{4} \in e_2$. Then, we have a two-peak solution which behaves

$$u_\epsilon(x) \sim \sum_{k=1}^{2} \frac{1}{12c} w(\frac{x - t_k}{\epsilon}), \quad v_\epsilon(t_k) \sim 12c \ (k = 1, 2)$$

with $t_k = t_k(\epsilon) \sim t_k^0$.

Remark 6 *In this case, the locations of two peaks feel the total length and the distance from the junction.*

Secondly, we consider the case in which two peaks locate on a same edge, e.g. e_1. Assume $l_1 > \frac{3}{4}L$. Let $t_1^0 := l_1 - \frac{L}{4} \in e_1, t_2^0 := l_1 - \frac{3}{4}L \in e_1$. Then, we have a two-peak solution which behaves

$$u_\epsilon(x) \sim \sum_{k=1}^{2} \frac{1}{12c} w(\frac{x - t_k}{\epsilon}), \quad v_\epsilon(t_k) \sim 12c \ (k = 1, 2)$$

with $t_k = t_k(\epsilon) \sim t_k^0$.

3.5 Basic strategy of the proof

First, for a given u we can solve the second equation

$$-Dv'' + \frac{c}{\epsilon} g(x) u^2 v = \frac{1}{L} \ (x \in \mathcal{G})$$

with the Kirchhoff condition at vertices. We denote by $v(x) := T[u](x)$. Since

$$D(T[u])'' = \frac{c}{\epsilon} g(x) u^2(x)(T[u])(x) - \frac{1}{L}.$$

So, we have

$$T[u](x) - \frac{1}{L}\int_{\mathcal{G}} T[u]\, ds = \int_{\mathcal{G}} G(x,y)\left\{\frac{c}{\epsilon} g(y) u^2(y)(T[u](y))\right\} dy.$$

So, the problem is reduced to find a solution $u \in H^2_{KN}(\mathcal{G})$ of

$$S(u) := -\epsilon^2 u''(x) + ug(x)u^2(x)(T[u])(x) = 0 \quad (x \in \mathcal{G}).$$

We will find a solution in the form

$$u(x) = \frac{1}{6c} w\left(\frac{x-t}{\epsilon}\right)\chi\left(\frac{x-t}{r_0}\right) + \phi(x)$$

with $|t - t^0| \le C\epsilon^{\frac{3}{4}}$ and

$$v(x) \sim \xi(t) = \frac{6c}{g(t)}.$$

Then, actually,

$$v(x) = T[w_{\epsilon,t} + \phi](x).$$

Now, using

$$T[w_{\epsilon,t} + \phi] = T[w_{\epsilon,t}] + \langle T'(w_{\epsilon,t}), \phi\rangle + N(\phi),$$

we arrive at

$$\begin{aligned} & S(w_{\epsilon,t} + \phi) \\ := \; & -\epsilon^2\phi'' + \phi - 2gw_{\epsilon,t}T[w_{\epsilon,t}]\phi - gw_{\epsilon,t}^2\langle T'(w_{\epsilon,t}), \phi\rangle + R_\epsilon + N_1(\phi) = 0, \end{aligned}$$

where $R_\epsilon := -\epsilon^2 w''_{\epsilon,t} + w_{\epsilon,t} - g(x)T[w_{\epsilon,t}]w^2_{\epsilon,t}$ and $N_1(\phi)$ is a higher order term of ϕ. Now, we consider the equation for $\overline{z}(y) := z(\epsilon y)$ fo a function $z(x)$. Thus,

$$\begin{aligned} & \overline{S(w_{\epsilon,t} + \phi)} \\ := \; & -\overline{\phi}'' + \overline{\phi} - 2\overline{gw_{\epsilon,t}}\,\overline{T[w_{\epsilon,t}]}\,\overline{\phi} - \overline{gw_{\epsilon,t}}^2\,\overline{\langle T'(w_{\epsilon,t}), \phi\rangle} + \overline{R_\epsilon} + \overline{N_1(\phi)} = 0. \end{aligned}$$

Let

$$L_{\epsilon,t}\overline{\phi} := -\overline{\phi}'' + \overline{\phi} - 2\overline{gw_{\epsilon,t}}\,\overline{T[w_{\epsilon,t}]}\,\overline{\phi} - \overline{gw_{\epsilon,t}}^2\,\overline{\langle T'(w_{\epsilon,t}), \phi\rangle}.$$

Then, we have

$$L_{\epsilon,t}\overline{\phi} + \overline{R_\epsilon} + \overline{N_1(\phi)} = 0.$$

Let $\Pi^\perp_{\epsilon,t}$ be the projection $L^2(\mathcal{G}_\epsilon)$ to

$$C^\perp_{\epsilon,t} := \{f \in L^2(\mathcal{G}_\epsilon) \mid \int_{\mathcal{G}_\epsilon} f\overline{w_{\epsilon,t}}'\, dy = 0\}.$$

We first solve

$$\Pi^{\perp}_{\epsilon,t} \circ \left(L_{\epsilon,t}\overline{\phi} + \overline{R_\epsilon} + \overline{N_1(\phi)} \right) = 0$$

on $\phi \in B(C_0\epsilon) \cap K^{\perp}_{\epsilon,t}$ for a suitable C_0, where

$$B(C_0\epsilon) := \{u \in H^2_{KN}(\mathcal{G}_\epsilon) \mid \|\overline{\phi}\|_{H^2(\mathcal{G}_\epsilon)} \le C_0\epsilon\},$$

$$K^{\perp}_{\epsilon,t} := \{u \in H^2_{KN}(\mathcal{G}_\epsilon) \mid \int_{\mathcal{G}_\epsilon} f\overline{w_{\epsilon,t}}'\,dy = 0\}.$$

• **Claim:** $L^{\perp}_{\epsilon,t} := \Pi^{\perp}_{\epsilon,t} \circ L_{\epsilon,t}$ is a bijection from $K^{\perp}_{\epsilon,t}$ to $C^{\perp}_{\epsilon,t}$. Moreover, there exists a constant $\lambda > 0$ such that

$$\|L^{\perp}_{\epsilon,t}\overline{\phi}\|_{L^2(\mathcal{G}_\epsilon)} \ge \lambda\|\overline{\phi}\|_{H^2(\mathcal{G}_\epsilon)} \quad (\forall\overline{\phi} \in K^{\perp}_{\epsilon,t}).$$

We note $\|R_\epsilon\|_{L^2(\mathcal{G}_\epsilon)} \le C_1\epsilon$ for some $C_1 > 0$. Then, by using the contraction mapping principle, for a suitable C_0 such that $C_0 > \frac{2C_1}{\lambda}$, we have a unique solution $\phi_{\epsilon,t} \in B(C_0\epsilon) \cap K^{\perp}_{\epsilon,t}$ of

$$\overline{\phi} = -(L^{\perp}_{\epsilon,t})^{-1}\{\overline{R_\epsilon} + \overline{N_1(\phi)}\} := M(\overline{\phi}).$$

Finally, determine t in the region $|t - t^0| \le C\epsilon^{\frac{3}{4}}$ so that

$$W(t) := \frac{1}{\epsilon}\int_{\mathcal{G}_\epsilon} \overline{S(w_{\epsilon,t} + \phi_{\epsilon,t})}\xi(t)\overline{w_{\epsilon,t}}'\,dy = 0.$$

Recall

$$\overline{S(w_{\epsilon,t} + \phi_{\epsilon,t})} = -\overline{w_{\epsilon,t}}'' + \overline{w_{\epsilon,t}} - \overline{g}\overline{w_2\epsilon,t}^2\overline{T(x_{\epsilon,t})} + L_{\epsilon,t}(\overline{\phi_{\epsilon,t}}) + \overline{N_1(\phi_{\epsilon,t})}.$$

Here, we have

$$\frac{1}{\epsilon}\left|\int_{\mathcal{G}_\epsilon} L_{\epsilon,t}(\overline{\phi_{\epsilon,t}})\overline{w_{\epsilon,t}}'\,dy\right| \le \frac{1}{\epsilon} \times O(\epsilon\|\overline{\phi_{\epsilon,t}}\|_{L^2}) \le O(\epsilon)$$

and

$$\frac{1}{\epsilon}\left|\int_{\mathcal{G}_\epsilon} \overline{N_1(\phi_{\epsilon,t})}\overline{w_{\epsilon,t}}'\,dy\right| \le O(\epsilon).$$

We also note that, since $\tilde{w_{\epsilon,t}}(y) = w_{\epsilon,t}(t + \epsilon y) = \frac{1}{6c}w(y)\chi(\frac{\epsilon y}{r_0})$ is an even function and $\tilde{w_{\epsilon,t}}'(y)$ is an odd function, we have

$$\frac{1}{\epsilon}\int_{\mathcal{G}_\epsilon}(-\overline{w_{\epsilon,t}}'' + \overline{w_{\epsilon,t}})\overline{w_{\epsilon,t}}'\,dy = \int_{\mathcal{G}}(-w''_{\epsilon,t} + w_{\epsilon,t})w'_{\epsilon,t}\,dx = \frac{1}{\epsilon}\int_{\mathbf{R}}(-\tilde{w_{\epsilon,t}}'' + \tilde{w_{\epsilon,t}})\tilde{w_{\epsilon,t}}'\,dy = 0.$$

Thus we have

$$W(t) = -\frac{1}{\epsilon}\int_{\mathbf{R}} \tilde{g}\tilde{w_{\epsilon,t}}^2\widetilde{T[w_{\epsilon,t}]}\xi(t)\tilde{w_{\epsilon,t}}'\,dy + O(\epsilon).$$

Now, using

$$\tilde{g}(y) = g(t + \epsilon y) = g(t) + g'(t)\epsilon y + O(\epsilon^2 y^2),$$

we have

$$\begin{aligned} W(t) &= -\frac{g(t)\xi(t)}{\epsilon}\int_{\mathbf{R}}\left(\widetilde{T[w_{\epsilon,t}]}(y)-\widetilde{T[w_{\epsilon,t}]}(0)\right)\tilde{w_{\epsilon,t}}^2\tilde{w_{\epsilon,t}}'\,dy \\ &- g'(t)\xi(t)\int_{\mathbf{R}} y\widetilde{T[w_{\epsilon,t}]}(y)\tilde{w_{\epsilon,t}}^2\tilde{w_{\epsilon,t}}'\,dy + O(\epsilon). \end{aligned}$$

Here, we also need the following key estimates:

$$\widetilde{T[w_{\epsilon,t}]}(y)-\widetilde{T[w_{\epsilon,t}]}(0)=\epsilon m(t)y+K(y)+O(y^2\epsilon^2)$$

and

$$\widetilde{T[w_{\epsilon,t}]}(y)=\xi(t)+O(\epsilon y).$$

We arrive at

$$W(t)=\frac{1}{(6c)^3}\left(-g(t)\xi(t)m(t)\int_{\mathbf{R}} yw^2(y)w'(y)\,dy-g((t)\xi(t)^2\int_{\mathbf{R}} yw^2(y)w'(y)\,dy\right)+O(\epsilon).$$

Using $\xi(t)=\frac{6c}{g(t)}$ and

$$\int_{\mathbf{R}} yw(y)^2w'(y)\,dy=\int_{\mathbf{R}}(\frac{w^3}{3})'\,dy=-\frac{1}{3}\int_{\mathbf{R}} w^3\,dy,$$

we get for some constant $C\neq 0$

$$W(t)=C(-m(t)-\frac{6cg'(t)}{g(t)^2})+O(\epsilon)=CF(t)+O(\epsilon)$$

for $|t-t^0|\leq C\epsilon^{\frac{3}{4}}$. So, there exists t_ϵ such that $|t_\epsilon-t_0|\leq C\epsilon^{\frac{3}{4}}$ with $W(t_\epsilon)=0$. This completes the proof.

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